Hermitian K-theory of the Gaussian 2-integers

by

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Abstract

In this thesis we the calculate the 2-completed homotopy type of the hermitian K-theory of the ring $\mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ of Gaussian 2-integers. Motivation and background from number theory, algebra and algebraic geometry are also provided. We give introductions to classical K-theory and hermitian K-theory as well as the Quillen plus-construction. The hermitian K-groups of $\mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ are computed in degrees 0, 1 and 2 using these tools. For the homotopy type we use an étale version of hermitian K–theory, in the spirit of Dwyer-Friedlander, and construct a decomposition of the hermitian K-theory space of $\mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ into a product of spaces whose homotopy types are known. By work of Berrick, Karoubi and Østvær the étale version of hermitian K-theory of $\mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ is 2-adically equivalent to ordinary hermitian K-theory.

Contents

Introduction

What is this thing called K–theory?

In mathematics one often encounters categories with some kind of sum operation on the objects. Examples include the category of finite sets $FSet$ with disjoint union $\coprod$ and the category $Vect_F$ of finite-dimensional vector spaces over a fixed field F with direct sum $\oplus$. For such a category $\mathscr{C}$ with sum the (isomorphism classes of) objects form a semi-group with respect to the sum operation. We can group complete this semi-group to get the group $K_0(\mathscr{C})$ called the Grothendieck group of $\mathscr{C}$. This group can tell us a lot about the additive structure on the objects of $\mathscr{C}$, but it forgets most of the morphism structure. In particular it forgets the many ways in which an object can be isomorphic to itself. Finite sets are classified by how many elements they have and finite dimensional vector spaces by their dimension. Using this it is not difficult to see that the groups $K_0(FSet)$ and $K_0(Vect_F)$ are both isomorphic to $\mathbb{Z}$. On the other hand the categories $FSet$ and $Vect_F$ are quite different. The main difference lies in the symmetries of the objects. A set with n elements has automorphism group isomorphic to the symmetric group Σ_n while the group of automorphisms of an n-dimensional F-vector space is isomorphic to $GL_n(F)$.

There are several completely general ways to group complete categories $\mathscr{C}$ with sum while taking the symmetries of the objects into account. These constructions all produce so-called infinite loop spaces and they are all essentially equivalent. By abuse of language we will speak about *the* resulting space $K(\mathscr{C})$, which is called the algebraic K-theory space of $\mathscr{C}$. The set $\pi_0(K(\mathscr{C}))$ comes with a natural group structure and there is a group isomorphism $K_0(\mathscr{C}) \cong \pi_0(K(\mathscr{C}))$. For $n \geq 0$ define the n-th K-group of $\mathscr{C}$ by $K_n(\mathscr{C}) = \pi_n(K(\mathscr{C}))$.

K-theory of rings

The algebraic K-theory of a ring A is obtained by applying the above constructions to the category $\mathscr{P}(A)$ of finitely generated projective modules with direct sum $\oplus$ as sum operation. The resulting algebraic K-theory space of A is denoted by $K(A)$. The algebraic K-theory of rings of integers $\mathcal{O}_F$ in number fields has been much studied over the years. The groups $K_n(\mathcal{O}_F)$ contain deep number theoretic information can be very difficult to compute. Quillen conjectured that for a large class of rings A and for odd regular primes ℓ there should be a spectral sequence of ℓ-adic étale cohomology groups of A converging to ℓ-completed K-groups of A. This is known as the Quillen-Lichtenbaum conjecture and was made precise by Dwyer and Friedlander. They defined an ℓ-adic étale K-theory [16] $K^{\text{ét}}(A[\frac{1}{\ell}])_\ell$ which has the desired cohomology spectral sequence converging to it, at least with certain restrictions on the ring A. This also works for $\ell = 2$ when A contains a square root of -1. There is a natural comparison map $K(A[\frac{1}{\ell}]) \to K^{\text{ét}}(A[\frac{1}{\ell}])$ and Dwyer and Friedlander reformulated the Quillen-Lichtenbaum conjecture as saying that this map induces isomorphisms on homotopy groups after ℓ-adic completion $K_n(A[\frac{1}{\ell}]) \otimes \mathbb{Z}_\ell \xrightarrow{\cong} K_n^{\text{ét}}(A[\frac{1}{\ell}])_\ell$. For the prime 2 it is known that the map from K-theory to étale K-theory is 2-adic equivalence for rings of integers in number fields by work of Rognes and Weibel and for more general rings from work of Østvær.

Hermitian K–theory

Parallel to the story of algebraic K-theory of rings one has hermitian K-theory of rings. The starting point is a ring A with an involution on it (often trivial). For an element $a \in A$ we write $\bar{a}$ for the image of a under the involution. One applies K-theory to the category of finitely generated projective A-modules with ε-hermitian forms on them. Here ε is an element of A, often equal to ± 1, satisfying $\varepsilon\bar{\varepsilon} = 1$. The forms can satisfy various symmetry and anti-symmetry conditions, encoded in the choice of ε, but must be compatible with the involution (see Section 2). The space ${}_\varepsilon KQ(A)$ one gets after applying K-theory is called the ε-hermitian K-theory space of A.

Berrick and Karoubi showed in [10] that there is a commuting square of hermitian K-theory spaces

$$\begin{array}{ccc} {}_\varepsilon KQ(\mathbb{Z}[\frac{1}{2}]) & \longrightarrow & {}_\varepsilon KQ^{top}(\mathbb{R}) \\ \downarrow & & \downarrow \\ {}_\varepsilon KQ(\mathbb{F}_3) & \longrightarrow & KQ^{top}(\mathbb{C}) \end{array}$$

that becomes homotopy cartesian after 2-adic completion. The superscript top means a topological version of hermitian K-theory. The analogous result is also true in algebraic K-theory and this K-theory square appears in the work of Bökstedt [13] and in the calculations of Dwyer and Friedlander in [17]. The goal for this thesis has been to show that the ε-hermitian K–theory space of the ring $\mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ with trivial involution is 2-adically equivalent to ${}_\varepsilon KQ(\mathbb{F}_5) \times \Omega({}_\varepsilon KQ^{top}(\mathbb{C}))$. This is shown in the last section using a hermitian analogue of étale K-theory which we call étale ε-hermitian K-theory. We construct a homotopy cartesian square of (étale) hermitian K-theory spaces

$$\begin{array}{ccc} {}_\varepsilon KQ^{\text{ét}}(\mathbb{Z}[\frac{1}{2}, \sqrt{-1}])_2 & \longrightarrow & {}_\varepsilon KQ^{top}(\mathbb{C})_2 \times (\Omega_\varepsilon KQ^{top}(\mathbb{C})_2) \\ \downarrow & & \downarrow \\ {}_\varepsilon KQ^{\text{ét}}(\mathbb{F}_5)_2 & \longrightarrow & {}_\varepsilon KQ^{top}(\mathbb{C})_2, \end{array}$$

where the subscript 2 denotes 2-completion. Then, by results from the article [11] by Berrick, Karoubi and Østvær showing that the comparison map ${}_\varepsilon KQ(\mathbb{Z}[\frac{1}{2}, \sqrt{-1}]) \otimes \mathbb{Z}_2 \to {}_\varepsilon KQ^{\text{ét}}(\mathbb{Z}[\frac{1}{2}, \sqrt{-1}])$ is an isomorphism for all $n \geq 1$, we have the desired result for ordinary hermitian K-theory. This result has a counterpart in algebraic K-theory in the work of Dwyer and Friedlander [17] and in the article [42]. The methods used here to show the result in étale hermitian K-theory are essentially translations from the algebraic case to the hermitian case.

Organization of the sections

The sections are organized as follows: In section 1 we begin by recalling some fundamental notions from algebraic number theory and commutative algebra which will be needed in the later sections. Then in section 2 we set up a theoretical framework for modules with hermitian forms and the corresponding notion for vector bundles. Section 3 takes us through the classical constructions in K-theory; K_0, K_1 and K_2 and some applications to vector bundles. In section 4 the higher algebraic and hermitian K-groups are introduced via the plus-construction. Some computational tools for higher hermitian K-groups are introduced and applied to the Gaussian 2-integers. Next, section 5 introduces methods from étale cohomology which will be needed in later computations. Section 6 begins with an introduction to simplicial schemes, then gives a non-technical outline of étale homotopy and defines étale K-theories. The next section 7 is the most technical. We compute the hermitian K-theory of the Gaussian 2-integers and some related rings in terms of the hermitian K-theory of a finite field and the topological hermitian K-theory of the complex numbers. An appendix A then explains some methods involving pro-objects that were applied in the computations of section 7.

The first three sections are meant to be readable for someone with a basic understanding of ring and module theory, homological algebra, category theory and topology. Section 4 relies more on topology, in particular when we start talking about spectra. For Section 5 one should have an understanding of scheme theory and sheaf cohomology. Sections 6 and 7 use simplicial techniques and assume knowledge of group schemes. Some machinery for spectral sequence and spectra is also used.

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1 Number theoretic preliminaries

First we collect some facts from number theory and commutative algebra. In this section all rings are assumed to be unital and commutative.

1.1 Dedekind domains

If A is a ring, then $A^\times$ denotes its group of multiplicative units.

Definition 1.1. Let A be an integral domain and K its field of fractions. A non-zero A-submodule $M \subseteq K$ is called a fractional ideal of A if there is an $a \in K^\times$ such that $aM \subseteq A$. A fractional ideal M is said to be invertible if there exists another fractional ideal N such that their product (in K) MN equals A.

Clearly any non-zero ideal $\mathfrak{a} \subseteq A$ is fractional.

Definition 1.2. Let A be a subring of the ring B. The integral closure of A in B consists of all $b \in B$ satisfying an equation of the form $b^n + a_1 b^{n-1} + \cdots + a_{n-1}b + a_n = 0$ for some $a_i \in A$. If A equals its integral closure in B it is said to be integrally closed in B.

Lemma 1.3: *Let C be the integral closure of A in B. Then C is a subring of B containing A.*

Proof. [2, p.60] □

Proposition 1.4: *Let A be a noetherian domain of dimension 1 with field of fractions K. The following are equivalent:*

1. *A is integrally closed in its fraction field.*
2. *For every prime ideal $\mathfrak{p} \neq (0)$ the local ring $A_\mathfrak{p}$ is a discrete valuation ring.*
3. *Every fractional ideal of A is invertible.*
4. *The fractional ideals of A form a group under multiplication in K.*
5. *Every non-zero ideal $\mathfrak{a}$ in A has a unique (up to order) factorization as a finite product of prime ideals $\mathfrak{a} = \prod_{i=1}^n \mathfrak{p_i}^{e_i}$.*

Proof. [2, ch. 9] □

Definition 1.5. A ring satisfying the above conditions is called a *Dedekind domain.*

Examples of Dedekind domains include $\mathbb{Z}$, $k[x]$ for k a field, discrete valuation rings, rings of regular functions on affine smooth curves over a field, and, as we will see below, rings of integers in number fields.

If A is a Dedekind domain with fraction field K, let $J(A)$ denote the group of fractional ideals of A. Any element a of $K^\times$ generates a fractional ideal aA. The rule $a \mapsto aA$ defines a group homomorphism div : $K^\times \to J(A)$ whose kernel is $A^\times$. A fractional ideal of the form aA is called principal and the image of div in $J(A)$ is denoted $P(A)$. The cokernel $J(A)/P(A)$ is called the ideal class group $Cl(A)$ (or Picard group $\mathrm{Pic}(A)$) of A. This can be summarized by an exact sequence of abelian groups:

$$1 \longrightarrow A^\times \longrightarrow K^\times \xrightarrow{\text{div}} J(A) \longrightarrow Cl(A) \longrightarrow 1$$

Proposition 1.6: *A Dedekind domain A is a unique factorization domain if and only if $Cl(A)$ is trivial, i.e., if and only if every fractional ideal is principal.*

Proposition 1.7: *Let A be a noetherian domain of dimension 1 and K its field of fractions. If K is a finite extension of K then the integral closure B of A in K is a Dedekind domain.*

Proof. [41, p.77] □

1.2 Rings of integers in number fields

Definition 1.8. A number field is a finite field extension of $\mathbb{Q}$. If K is a number field the integral closure of $\mathbb{Z}$ in K is called the ring of integers in K and is denoted by $\mathcal{O}_K$.

By Proposition 1.7 above $\mathcal{O}_K$ is a Dedekind domain.

Definition 1.9. Let S be a finite set of non-zero prime ideals of $\mathcal{O}_K$. The subring of K defined by $\mathcal{O}_K^S = \left\{\frac{f}{g} | f \in \mathcal{O}_K, g \in \mathfrak{p} \setminus \{0\} \text{ for some } \mathfrak{p} \in S\right\}$ is called the ring on S-integers of K.

If S is the set of primes occurring in the factorization of $n\mathcal{O}_K$ then $\mathcal{O}_K^S = \mathcal{O}_K[\frac{1}{n}]$ by definition. It is called the ring of n-integers of K. In this thesis the main objective is to study invariants of the ring $\mathbb{Z}[\sqrt{-1}, \frac{1}{2}]$ of Gaussian 2-integers.

Let L/K be a degree n-extension of number fields so that L is an n-dimensional K-vector space. Any element $x \in L$ defines a K-linear map $T_x : L \to L$ by multiplication. The *trace* of x is defined to be the trace of the map T_x

$$Tr_{L/K}(x) = \mathrm{tr}(T_x).$$

Similarly, the *norm* of x is defined by the determinant

$$N_{L/K}(x) = \det(T_x).$$

Remark 1.10. Let $\overline{K}$ be an algebraic closure of K and identify K with its image in $\overline{K}$. Since L/K is separable, the number of embeddings $L \hookrightarrow \overline{K}$ over K equals n. One has the following formulas

$$Tr_{L/K}(x) = \sum_{\sigma \in I_K(L)} \sigma(x)$$

and

$$N_{L/K}(x) = \prod_{\sigma \in I_K(L)} \sigma(x)$$

where $I_K(L) = \mathrm{Hom}_K(L, \overline{K})$. For details see [41, p.9].

If $\mathfrak{a}$ is a non-zero ideal of $\mathcal{O}_K$ the quotient ring $\mathcal{O}_K/\mathfrak{a}$ is finite. The *norm* of $\mathfrak{a}$ is defined by $N(\mathfrak{a}) = \#(\mathcal{O}_L/\mathfrak{a})$. If $\mathfrak{a} = (x)$ for some non-zero x in $\mathcal{O}_K$ then $N_{K/\mathbb{Q}}(x) = N(\mathfrak{a})$.

Any ring of integers $\mathcal{O}_K$ is a free, finitely generated $\mathbb{Z}$-module [41, p.12]. Let $\{\alpha_i\}$ be a basis for K over $\mathbb{Q}$. The discriminant $\Delta_K(\alpha_1, \ldots, \alpha_n)$ of $\{\alpha_i\}$ is defined to be the rational number $\det(Tr_{K/\mathbb{Q}}(\alpha_i \alpha_j))$. If one considers only bases which are also bases for $\mathcal{O}_K$ as a $\mathbb{Z}$-module, then this number turns out to be an integer independent of the choice of basis. The discriminant Δ_K of K is defined to be the discriminant of any such basis.

A real embedding of a number field K is an embedding $K \hookrightarrow \mathbb{R}$. Let r_K denote the number of distinct real embeddings of K. A complex embedding of K is an embedding $K \hookrightarrow \mathbb{C}$ which does not factor through $\mathbb{R}$. The complex embeddings appear in pairs of complex conjugates. Let c_K denote the number of pairs of complex embeddings of K. Note that if $K/\mathbb{Q}$ is an extension of degree n then

$$n = r_K + 2c_K$$

Now we can state a fundamental result in number theory.

Theorem 1.11: *(Minkowksi bound theorem) Let K be a number field of degree n over $\mathbb{Q}$. For any fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$ there is an ideal $\mathfrak{b} \subseteq \mathcal{O}_K$ and an $x \in K^\times$ such that $\mathfrak{a} = x\mathfrak{b}$ and $\mathfrak{b}$ satisfies:*

$$N(\mathfrak{b}) \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^{c_K} \sqrt{|\Delta_K|}$$

Proof. [41, p.38] ☐

This has an immediate consequence:

Corollary 1.12: *The class group $Cl(\mathcal{O}_K)$ is finite.*

The next result is also fundamental.

Theorem 1.13: *(Dirichlet's unit theorem) Let K be a number field with r_K real embeddings and c pairs of complex embeddings. There is an isomorphism*

$$(\mathcal{O}_K)^\times \approx \mu(K) \times \mathbb{Z}^{r_K + c_K - 1}$$

where $\mu(K)$ denotes the roots of unity in K.

Proof. [41, p.42] ☐

For the ring of S-integers $\mathcal{O}_K^S$, where S is a finite set of non-zero primes in $\mathcal{O}_K$, there is a natural surjection $Cl(\mathcal{O}_K) \twoheadrightarrow Cl(\mathcal{O}_K^S)$, so the latter group is also finite[41, p.70]. For the units there is an isomorphism of groups [41, p.71]

$$(\mathcal{O}_K^S)^\times \cong \mu(K) \times \mathbb{Z}^{r_K + c_K + \#S - 1}. \tag{1}$$

Let L/K be a finite extension of number fields and consider a non-zero prime ideal $\mathfrak{p} \subset \mathcal{O}_K$. The ideal $\mathfrak{p}\mathcal{O}_L \subset \mathcal{O}_L$ has a factorization into prime ideals $\mathfrak{p}\mathcal{O}_L = \prod_{i=1}^r \mathfrak{q}_i^{e_i}$. The number e_i is called the *ramification index* of $\mathfrak{q}_i$ over $\mathfrak{p}$. One is often interested in prime ideals $\mathfrak{p}$ for which $e_i = 1$ for all $1 \leq i \leq r$; such ideals are called *unramified.* The number of primes above $\mathfrak{p}$ in $\mathcal{O}_L$ is r. For each prime ideal $\mathfrak{q}_i$ over $\mathfrak{p}$ there is an induced finite field extension $\mathcal{O}_K/\mathfrak{p} = k(\mathfrak{p}) \subseteq k(\mathfrak{q}_i) = \mathcal{O}_L/\mathfrak{q}_i$ whose degree f_i is called the *inertia degree* of $\mathfrak{q}_i$ over $\mathfrak{p}$.

Theorem 1.14: *(Fundamental identity) Let L/K be a degree n-extension of number fields and $(0) \neq \mathfrak{p}$ a prime ideal of $\mathcal{O}_K$. If $\mathfrak{p}\mathcal{O}_K = \prod_{i=1}^r \mathfrak{q}_i^{e_i}$ and f_i is the inertia degree of $\mathfrak{q}_i$ one has:*

$$\sum_{i=1}^r e_i f_i = n$$

Proof. [41, p.46] □

Lemma 1.15: *Let K be a number field. A prime ideal $(0) \neq (p) \subset \mathbb{Z}$ ramifies in $\mathcal{O}_K$ if and only if p divides the discriminant Δ_K.*

Proof. [41, p.203] □

Let K be a number field different from $\mathbb{Q}$. Then the discriminant Δ_K of K is non-zero and by Minkowksi's theorem [41, p.207] it is different from ± 1. Therefore, for any number field K, some prime will ramify in $\mathcal{O}_K$. Certain algebraic constructions only work when the involved maps are unramified. A partial solution to this problem is to invert the discriminant Δ_K in all the involved rings. We will come back to this when we talk about étale cohomology in chapter 5

Example 1.16. (The Gaussian integers)[41, I.§1] Let $i = \sqrt{-1}$ and $K = \mathbb{Q}(i)$. Then $\mathcal{O}_K$ is $\mathbb{Z}[i]$, the ring of Gaussian integers. One has $2 = (1+i)(1-i) = (1+i)^2(-i)$ where i is a unit and $1+i$ is irreducible. It follows that $(2)\mathcal{O}_K = (1+i)^2$. Hence (2) is ramified in K with ramification index 2. It is a classical theorem of number theory that an odd prime p is congruent to 1 modulo 4 if and only if it is a sum of squares. If $p = a^2 + b^2$ it has the factorization $a^2+b^2 = (a+ib)(a-ib)$ in $\mathbb{Z}[i]$. A straightforward argument with norms shows that $(p) = (a+ib)(a-ib)$ gives a factorization of the ideal $p\mathbb{Z}[i]$ into prime ideals. The prime ideals in $\mathbb{Z}[i]$ are of the following three types[41, p.4]:

- $(1+i)$ which lies over (2)
- $(a+ib)$ $a, b \in \mathbb{Z}$, which lies over $p = a^2 + b^2$, where $p \equiv 1 \mod (4)$

- (p) for $p \in \mathbb{Z}$ such that $p \equiv 3 \mod (4)$

In particular all ideals of $\mathbb{Z}[i]$ are principal, hence $Cl(\mathbb{Z}[i]) = 0$. Another way to see this is by using the Minkowski bound (1.11). Here $n = 2$, $c = 1$ and $\Delta_K = -4$. Thus, for any prime ideal $\mathfrak{p}$ there is an $x \in \mathbb{Q}(i)^\times$ such that $N(\mathfrak{p}x^{-1}) < \frac{4}{\pi}$ which is less than 2. This can only happen if $N(\mathfrak{p}x^{-1}) = 1$ that is, if $\mathfrak{p}$ equals $x\mathbb{Z}[i]$. By Dirichlet's unit theorem the units in $\mathbb{Z}[i]$ are just the roots of unity $\mu(\mathbb{Q}(i)) = \{1, i, -1, -i\}$. Considering now the Gaussian 2-integers instead, Dirichlet's unit theorem for S-integers (1) gives an isomorphism

$$(\mathbb{Z}[i, \frac{1}{2}])^\times \cong \mathbb{Z} \times \mu_4$$

where the free summand corresponds to group of powers of $1 + i$ and the roots of unity in $\mathbb{Q}(i)$ are $\mu_4 = \{1, i, -1, -i\}$. Except for the prime $(1 + i)$, the ring $\mathcal{O}^2_{\mathbb{Q}(i)}$ has the same prime ideals as the Gaussian integers. Furthermore, since $\mathcal{O}_{\mathbb{Q}(i)}$ is Euclidean, and $\mathcal{O}^2_{\mathbb{Q}(i)}$ is a localization of $\mathcal{O}_{\mathbb{Q}(i)}$, the ring $\mathcal{O}^2_{\mathbb{Q}(i)}$ is also Euclidean [48, prop.7].

1.3 Absolute Galois groups

Let K be a field and consider the category $\mathscr{G}(K)$ of finite Galois extensions of K. A map in $\mathscr{G}(K)$ is an extension $L \to L'$ over K. For a finite Galois extension L/K let $\mathrm{Gal}(L/K)$ denote its Galois group. An extension $L \to L'$ over K induces a map $\mathrm{Gal}(L'/K) \to \mathrm{Gal}(L/K)$ by restriction, hence $\mathrm{Gal}(-)$ is a functor from $\mathscr{G}(K)^{op}$ to the category of groups. Choose a separable closure $\bar{K}$ of K and let $\mathrm{Gal}(\bar{K}/K)$ be its Galois group. It is called the absolute Galois group of K. The group $\mathrm{Gal}(\bar{K}/K)$ maps by restriction to the Galois groups of all finite Galois extensions of K. These maps induce an isomorphism [41, p.271]:

$$\mathrm{Gal}(\bar{K}/K) \xrightarrow{\cong} \varprojlim_L \mathrm{Gal}(L/K) = \lim_{L \in \mathscr{G}(K)^{op}} \mathrm{Gal}(L/K)$$

Example 1.17. Let $q = p^r$ for some prime p and natural number r and fix a separable closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$. Since any finite extension of a finite field is separable, $\bar{\mathbb{F}}_q$ is an algebraic closure of $\mathbb{F}_q$. To each natural number n there corresponds a unique subfield $\mathbb{F}_{q^n}$ of $\bar{\mathbb{F}}_q$ with q^n elements. It has extension degree n over $\mathbb{F}_q$ and any extension $\mathbb{F}_{p^r}/\mathbb{F}_{p^s}$ of finite fields is normal (see e.g. [50]). The Galois group $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is cyclic of order n, generated by the element φ given by $\varphi(x) = x^{q^n}$. This morphism is called the Frobenius element of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. If $n = kl$ the element $\varphi^k \in \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ generates a subgroup of order $\frac{n}{k} = l$ whose fixed field is $\mathbb{F}_{q^l}$. The map on Galois groups $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \to \mathrm{Gal}(\mathbb{F}_{q^l}/\mathbb{F}_q)$ corresponds to the quotient map $\mathbb{Z}/n \to \mathbb{Z}/l$. Taking the limit over all the groups gives a group isomorphic to $\hat{\mathbb{Z}}$, the pro-finite completion of the integers [41, p.272].

Example 1.18. The field $\mathbb{C}$ of complex numbers is algebraically closed, and hence has no non-trivial separable extensions. It follows that the absolute Galois group of $\mathbb{C}$ is trivial.

Example 1.19. The only non-trivial finite field extension of the reals $\mathbb{R}$ is $\mathbb{C}/\mathbb{R}$ which has Galois group isomorphic to $\mathbb{Z}/2$. The absolute Galois group of $\mathbb{R}$ is therefore isomorphic to $\mathbb{Z}/2$, where the non-trivial element acts by complex conjugation.

2 Hermitian modules and vector bundles

In this section we set up a general framework for hermitian forms on finitely generated projective modules over a ring. This theory encompasses symmetric, symplectic and hermitian forms and even works for non-commutative rings when they have an anti-involution. The aim is to give a theoretical background for Hermitian K-theory. In later sections we will only apply the theory to commutative rings with trivial involution, and what are usually called symmetric and symplectic forms. As an application we extend the theory to describe vector bundles with form.

There is an even more general framework described by A. Bak in the book [3] using so-called form parameters. Eventually we will make the assumption that 2 be a unit in every ring we work with and this can be avoided by using Bak's framework. However, treating Bak's theory would take us too far afield, so in stead we follow a middle road which is closer to the approach of e.g. [29].

2.1 Modules

In this chapter all rings are assumed to be unital but not necessarily commutative and all modules are right modules.

Definition 2.1. Let A be a ring. An anti-involution τ on A is an additive unital function $\tau : A \to A$ such that $\tau(ab) = \tau(b)\tau(a)$ for all $a, b \in A$ and $\tau \circ \tau = id_A$.

Definition 2.2. A hermitian ring is a pair (A, τ) where A is a ring and τ is an anti-involution on A. To simplify notation we write $\bar{a}$ for $\tau(a)$ and often write simply A for (A, τ). A hermitian morphism $f : (A, \tau_A) \to (B, \tau_B)$ is a ring homomorphism $f : A \to B$ such that $f(\bar{a}) = \overline{f(a)}$ for all $a \in A$.

The canonical examples of hermitian rings are $\mathbb{C}$ with complex conjugation, $M_{n,n}(\mathbb{C})$ with conjugate transpose and group rings $\mathbb{Z}[G]$ with $g \mapsto g^{-1}$. Note that for any commutative ring A, the pair (A, id_A) is a hermitian ring.

Definition 2.3. Let A be a hermitian ring and M an A-module. The hermitian dual M^* of M is the group of $\mathbb{Z}$-linear maps $f : M \to A$ such that $f(ma) = \bar{a}f(m)$ for all $m \in M$ and $a \in A$.

We give M^* the structure of a right A-module by setting $(fa)(m) = f(m)a$ where $m \in M$, $f \in M^*$ and $a \in A$. If M is a finitely generated projective module, then so is M^*. An A-linear map $\alpha : M \to N$ induces a map $\alpha^* : N^* \to M^*$ given by $\alpha^*(f) = f \circ \alpha$. This defines an endofunctor on the category $\mathscr{P}(A)$ of finitely generated projective A-modules. The following lemma is easily verified.

Lemma 2.4: *For any A-module M in $\mathscr{P}(A)$ the hermitian evaluation map $ev_M : M \to (M^*)^*$ given by $m \mapsto (f \mapsto \overline{f(m)})$ is a natural isomorphism of A-modules.*

Note that Lemma 2.4 is false if M is not finitely generated.

Definition 2.5. Let A be hermitian ring and ε an element of the center of A such that $\varepsilon\bar{\varepsilon} = 1$. An ε-hermitian module over A is an A-module M in $\mathscr{P}(A)$ with a $\mathbb{Z}$-bilinear map $\varphi : M \times M \to A$ satisfying

$$\begin{aligned} \varphi(ma, m'b) &= \bar{a}\varphi(m, m')b \\ \varphi(m, m') &= \varepsilon\overline{(\varphi(m', m))} \end{aligned}$$

for $m, m' \in M$, $a, b \in A$. A form with these properties is called ε-hermitian and we write (M, φ) for such an ε-hermitian module.

Example 2.6. The vector space $\mathbb{C}^n$ with the hermitian form

$$\big((z_1, \ldots, z_n), (z'_1, \ldots, z'_n)\big) \mapsto \sum_{i=1}^{n} \bar{z}_i z'_i$$

is a 1-hermitian $(\mathbb{C}, \tau)$-module, where τ denotes complex conjugation. Note that here the left variable is conjugated, while in the standard hermitian inner product the right variable in conjugated.

Remark 2.7. Let A be a commutative hermitian ring with trivial involution. A 1-hermitian form on an A-module M is what is usually called a *symmetric bilinear form* on M. When $char(A) \neq 2$ a -1-hermitian form on M is usually called a *symplectic form* on M.

Any hermitian form φ on A^n has a matrix representation $\varphi(v, w) = \bar{v}^T[\varphi]w$, where $[\varphi]_{ij} = \varphi(e_i, e_j)$ and v, w are any two column vectors in A^n. We will sometimes write (A^n, P), with $P \in M_{n,n}(A)$, when we mean the hermitian module whose representing matrix is P. From the equation

$$\varphi(m, m') = \varepsilon\overline{(\varphi(m', m))}$$

it follows that any matrix P representing an ε-hermitian form must satisfy $P = \varepsilon\bar{P}^T$. Matrices satisfying this equation are called ε-hermitian. The representing matrix in Example 2.6 is the $n \times n$ identity matrix.

Remark 2.8. Giving an ε-hermitian form φ on a finitely generated projective module M is equivalent to giving a morphism $\tilde{\varphi} : M \to M^*$ such that the composite $M \overset{ev_M}{\to} M^{**} \overset{\tilde{\varphi}^*}{\to} M^*$ equals $\overline{\tilde{\varphi}}\varepsilon$. Here $\overline{\tilde{\varphi}}$ is given by $\overline{\tilde{\varphi}}(m)(m') = \overline{\tilde{\varphi}(m)(m')}$ which defines an element of M^* because of the condition on $\tilde{\varphi}$.

Definition 2.9. A hermitian form φ is called non-degenerate if the associated map $\tilde{\varphi} : M \to M^*$ is an isomorphism. A hermitian module is called non-degenerate if its form is non-degenerate.

Definition 2.10. Let (M, φ_M) and (N, φ_N) be hermitian modules. An isometry

$$f : (M, \varphi_M) \to (N, \varphi_N)$$

is an A-linear map $f : M \to N$ which preserves the hermitian form, i.e. $\varphi_N(f(m), f(m')) = \varphi_M(m, m')$ for all $m, m' \in M$.

This can also be stated using the associated maps (see (2.8) above) to the hermitian duals. The map f is an isometry if and only if the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\tilde{\varphi}_M"] \arrow[d, "f"] & M^* \\ N \arrow[r, "\tilde{\varphi}_N"] & N^*. \arrow[u, "f^*"'] \end{tikzcd}$$

The identity map is clearly an isometry and a composition of isometries is again an isometry. Thus (for a fixed value of ε) the non-degenerate ε-hermitian modules over A and the isometries between them form a category, denoted by ${}_\varepsilon\mathscr{Q}(A)$. Note that an isometry $f : (M, \varphi_M) \to (N, \varphi_N)$ with φ_M non-degenerate is necessarily injective.

Definition 2.11. The *orthogonal sum* $(M, \varphi) \perp (N, \psi)$ of the hermitian modules (M, φ) and (N, ψ), has $M \oplus N$ as underlying module and has the form $\varphi \perp \psi$ which sends $\big((m, n), (m', n')\big)$ to $\varphi(m, m') + \psi(n, n')$.

The forms φ and ψ are non-degenerate if and only if $\varphi \perp \psi$ is. Observe that the orthogonal sum is *not* the categorical coproduct in ${}_\varepsilon\mathscr{Q}(A)$; if this were the case then the identity map on (M, φ) would induce a map $(M, \varphi) \perp (M, \varphi) \to (M, \varphi)$, but this in only injective when $M = 0$. For similar reasons it is not the categorical product either.

Definition 2.12. Let A be a commutative hermitian ring and let (M, φ), (N, ψ) be 1-hermitian modules. Their tensor product $(M, \varphi) \otimes (N, \psi)$ has as underlying module the usual tensor product $M \otimes_A N$ and the form given by

$$(\varphi \otimes \psi)(m \otimes n, m' \otimes n') = \varphi(m, m') \cdot \psi(n, n').$$

The form $\varphi \otimes \psi$ is non-degenerate if and only if each of the factors is.

Definition 2.13. Let M be an A-module in $\mathscr{P}(A)$. The *hyperbolic module* ${}_\varepsilon H(M)$ of M has underlying module $M \oplus M^*$ and the form given by

$$\varphi((m, f), (m', g)) = \overline{f(m')} + \varepsilon g(m).$$

If $M \xrightarrow{\alpha} N$ is an isomorphism, let

$${}_\varepsilon H(\alpha) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & (\alpha^{-1})^* \end{smallmatrix}\right) : M \oplus M^* \to N \oplus N^*.$$

This defines a functor ${}_\varepsilon H : iso(\mathscr{P}(A)) \to iso({}_\varepsilon\mathscr{Q}(A))$, where for a category $\mathscr{C}$, $iso(\mathscr{C})$ denotes the groupoid of isomorphisms in $\mathscr{C}$.

In the other direction there is a forgetful functor $U : {}_\varepsilon\mathscr{Q}(A) \to \mathscr{P}(A)$ which sends a hermitian module to its underlying module. The following lemma relating these two functors is fundamental for the comparison of algebraic and hermitian K-theory.

Lemma 2.14: *[31, p.62] Let (A,τ) be a hermitian ring such that 2 is a unit in A, and let (M,φ) be a non-degenerate ε-hermitian A-module. There is a natural isomorphism*

$$(M,\varphi) \perp (M,-\varphi) \xrightarrow{\cong} {}_\varepsilon H(M)$$

Proof. Let $\tilde{\varphi} : M \to M^*$ be the map associated to the form φ. It sends $m' \in M$ to $(m \mapsto \varphi(m,m'))$, and is an isomorphism by assumption. The map $\eta_{(M,\varphi)} : M \oplus M \to M \oplus M^*$ given in matrix notation by

$$\eta_{(M,\varphi)} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \tilde{\varphi} & \tilde{\varphi} \end{pmatrix}$$

is a linear isomorphism with inverse $\begin{pmatrix} 1 & \frac{1}{2}\tilde{\varphi}^{-1} \\ -1 & \frac{1}{2}\tilde{\varphi}^{-1} \end{pmatrix}$. A straightforward computation shows that the forms are also preserved and that the morphism is natural in (M,φ). □

Definition 2.15. Let (M,φ) be an ε-hermitian A-module. Its automorphism group is called the orthogonal group of (M,φ) and is denoted $O(M,\varphi)$.

The hyperbolic module ${}_\varepsilon H(A^n)$ on A^n is called hyperbolic $2n$-space (plane if $n = 1$). Its orthogonal group is called the n-th ε-orthogonal group and is denoted ${}_\varepsilon O_{n,n}(A)$. By taking the standard basis for A^n and the dual basis for $(A^n)^*$ the group ${}_\varepsilon O_{n,n}(A)$ can be identified [29, p.40] with the subgroup of $GL_{2n}(A)$ consisting of the matrices

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad P^{-1} = \begin{pmatrix} \bar{d}^T & \varepsilon\bar{b}^T \\ \bar{\varepsilon}\bar{c}^T & \bar{a}^T \end{pmatrix}.$$

Remark 2.16. When A is commutative hermitian ring with trivial involution the group ${}_{-1}O_{n,n}(A)$ is usually denoted by $Sp_{2n}(A)$ and called the n-th symplectic group of A. The group ${}_1O_{n,n}(A)$ is often called the n-th orthogonal group of A and denoted by $O_{2n}(A)$.

Lemma 2.17: *Let (M,φ) be an ε-hermitian module over (A,τ), where 2 is a unit in A. Then, for some n, (M,φ) is an orthogonal summand of ${}_\varepsilon H(A^n)$.*

Proof. Since M is in $\mathscr{P}(A)$ there is an N in $\mathscr{P}(A)$ and some number n such that $M \oplus N$ is isomorphic to A^n. We have isomorphisms ${}_\varepsilon H(M \oplus N) \cong {}_\varepsilon H(A^n)$ and ${}_\varepsilon H(M \oplus N) \cong {}_\varepsilon H(M) \perp {}_\varepsilon H(N)$. By Lemma 2.14 we have an isomorphism ${}_\varepsilon H(M) \cong (M,\varphi) \perp (M,-\varphi)$ and the result follows. □

As a consequence every orthogonal group $O(M,\varphi)$ injects (non-canonically) into ${}_\varepsilon O_{n,n}(A,\tau)$ for some n.

Example 2.18. Let (A, τ) be a hermitian ring and let a be an element of A such that $\bar{a} = a$. Define $\langle a \rangle_A$ to be the 1-hermitian module (A, μ_a) where $\mu_a(b, c) = \bar{b}ac$ for a and b in A. The form is non-degenerate if and only if a is a unit. If $a = \bar{c}bc$ then left multiplication by c defines an isometry $\langle a \rangle_A \xrightarrow{c\cdot} \langle b \rangle_A$ which is an isomorphism if and only if c is a unit.

Lemma 2.19: *Let A be a commutative local hermitian ring with trivial involution. If 2 is a unit of A, then for every non-degenerate 1-hermitian A-module (M, φ) there are units $u_1, \ldots, u_n$ of A such that*

$$(M, \varphi) \cong \langle u_1 \rangle_A \perp \cdots \perp \langle u_m \rangle_A.$$

Proof. [39, p.6] □

The units u_i are not uniquely determined by M and φ.

Lemma 2.20: *Let A be a commutative ring with trivial involution. If A is a Dedekind domain or a local ring every non-degenerate (-1)-hermitian A-module is isomorphic to ${}_{-1}H(A^n)$ for some n.*

Proof. [39, p.7] □

We end the section by examining the ε-orthogonal groups over $\mathbb{R}$ and $\mathbb{C}$ with trivial involutions. The homotopy types of the groups, taken from [30, 9.23], are included because they will be needed when we consider hermitian vector bundles later. Note that the notation here differs slightly from the notation in Remark 2.7.

Example 2.21. Let J_ε denote the matrix $\left(\begin{smallmatrix} 0 & \varepsilon I \\ I & 0 \end{smallmatrix}\right)$ where I denotes the $n \times n$ identity matrix.

- The group ${}_1O_{n,n}(\mathbb{C})$ consists of the $2n \times 2n$ invertible complex matrices P such that $P^T J_1 P = J_1$. The factorization

  $$\begin{pmatrix} 1 & \frac{i}{2} \\ 1 & -\frac{i}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{i}{2} & -\frac{i}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  shows that the map given by $v \mapsto \left(\begin{smallmatrix} 1 & 1 \\ i\frac{1}{2} & -i\frac{1}{2} \end{smallmatrix}\right) v$ is an isomorphism

  $$\left(\mathbb{C}^{2n}, \left(\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right)\right) \xrightarrow{\cong} \left(\mathbb{C}^n \oplus \mathbb{C}^n, \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)\right)$$

  of 1-hermitian modules. The orthogonal group of $\left(\mathbb{C}^{2n}, \left(\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right)\right)$ is known from Lie group theory. It is denoted by $O_{2n}(\mathbb{C})$ and called the group of complex orthogonal matrices. When given the usual topology it contains the group $O(2n)$ of real orthogonal $2n \times 2n$-matrices as a deformation retract.

- For the case $\varepsilon = -1$ the hermitian form is given by the matrix $J_{-1} = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. The matrices respecting this form are the same ones that respect the form given by $-J_{-1}$ and these are by definition the complex symplectic matrices $Sp_{2n}(\mathbb{C})$. There is a homotopy equivalence $Sp_{2n}(\mathbb{C}) \simeq Sp(n)$ where the latter is the group of isometries of quaternionic n-space with the usual hermitian form.

- Similar considerations for $\mathbb{R}$ with trivial involution give ${}_1O_{n,n}(\mathbb{R}) \cong O_{n,n}(\mathbb{R})$ [30, 9.23] where the latter is homotopy equivalent to $O(n) \times O(n)$, and ${}_{-1}O_{n,n}(\mathbb{R}) = Sp_{2n}(\mathbb{R})$ which has the homotopy type of $U(n)$.

2.2 Vector bundles

Definition 2.22. Let $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ with the usual topology. An F-vector bundle consists of the following data:

- A triple (E, p, B) where E and B are topological spaces and $p : E \to B$ is a continuous surjective map.

- For each $b \in B$ the structure of a finite dimensional F-vector space on the fiber $p^{-1}(\{b\})$.

- The existence of a covering $\{U_i\}$ of B and homeomorphisms $\tau_i : p^{-1}(U_i) \xrightarrow{\cong} U_i \times F^n$ over U_i, which are linear in each fiber.

The space E is called the total space of the bundle, B is called the base space and p is called the projection map. The maps τ_i are called local trivializations or charts. We will often write simply $p : E \to B$ for a bundle or even E when the base space and projection are understood.

For a fixed space X the F-vector bundles with base space X, called vector bundles *over* X, form a category $\mathscr{V}_F(X)$. The morphisms in $\mathscr{V}_F(X)$ are continuous functions $E \to E'$ which are linear on fibers and compatible with projections to X. Many functorial constructions on vector spaces have analogs for vector bundles. The category $\mathscr{V}_F(X)$ is additive under the operation Whitney sum, which is a fiberwise direct sum of vector bundles. Furthermore the operations tensor product, dual and exterior powers are defined for vector bundles [30, p.18].

If $p : E \to B$ is a bundle and $f : X \to B$ is a continuous map there is an induced bundle $f^*(E) \to X$ called the pullback bundle of E along f. Vector bundles on X which are pulled back along homotopic maps are isomorphic [30, p.35]. The isomorphisms classes of vector bundles over a fixed space X form a set, so we get a functor $[\mathscr{V}_F] : HoTop^{op} \to Set$ which sending a space X to the set $[\mathscr{V}_F](X)$ of isomorphism classes of F-vector bundles over X. If $f : X \to Y$ is a continuous map $[\mathscr{V}_F](f) : [\mathscr{V}_F](Y) \to [\mathscr{V}_F](X)$ sends a class $[E]$ of bundles on Y to the class $[f^*(E)]$ of bundles on X.

Let $C_F(X)$ denote the ring of continuous F-valued functions on X. For a given F-vector bundle $p : E \to X$ the continuous sections $\Gamma(X, E)$ of p form a $C_F(X)$-module under pointwise multiplication.

Theorem 2.23: *(Serre-Swan) Let X be a compact Hausdorff space and $A = C_F(X)$. Then for any vector bundle $p : E \to X$ the A-module $\Gamma(X, E)$ is finitely generated projective and the functor $\Gamma(X, -) : \mathscr{V}_F(X) \to \mathscr{P}(A)$ taking a bundle $p : E \to X$ to the module $\Gamma(X, E)$ is an equivalence of categories.*

Proof. [30, p.32] □

Now let $\varepsilon = \pm 1$ and fix a continuous involution τ of F. We get an *ε-hermitian* F-vector bundle on a space X by modifying Definition 2.22 above in the following way: The fibers $p^{-1}(b)$ should have the structure of ε-hermitian F-vector spaces and the trivializations take the form $\tau_i : p^{-1}(U_i) \xrightarrow{\cong} U_i \times (F^n, \varphi_i)$ where φ_i is an ε-hermitian form on F^n and the τ_is are isometries on each fiber.

The hermitian duality functor on the category $Vect_F$ of F-vector spaces extends to the category of vector bundles over any space, by taking hermitian duals fiberwise [30, p.20]. Just as for modules there is a natural isomorphism $ev_E : E \to (E^*)^*$ and giving an ε-hermitian form φ on a vector bundle E is equivalent to giving a bundle map $\tilde{\varphi} : E \to E^*$ such that $\tilde{\varphi}^* \circ ev_E = \overline{\tilde{\varphi}}\varepsilon$. An ε-hermitian form is non-degenerate if the associated bundle map is an isomorphism. Further, there are obvious generalizations of the notions of tensor product and orthogonal sum of ε-hermitian vector bundles. The ε-hermitian F-vector bundles over X form a category ${}_\varepsilon\mathscr{H}_{(F,\tau)}(X)$ with fiberwise isometries as maps.

Lemma 2.24: *Let X be a normal space and $p : E \to X$ an F-vector bundle. There is a natural isomorphism of $C_F(X)$-modules*

$$f_{X,E} : \Gamma(X, E^*) \xrightarrow{\cong} \Gamma(X, E)^*$$

which sends a section $s \in \Gamma(X, E^*)$ to the homomorphism which sends a section $t \in \Gamma(X, E)$ to the function $x \mapsto s(x)(t(x)) \in F$.

Proof. The key ingredient in the proof is the fact that, for any point $x \in X$ and vector $v_x \in p^{-1}(\{x\})$ there is a section $v \in \Gamma(X, E)$ with $v(x) = v_x$. This follows from the Tietze extension theorem since X is normal [40, p.219].

We construct an inverse $t_{X,E}$ to $f_{X,E}$ as follows. For a homomorphism $\lambda : \Gamma(X, E) \to F$ and $v_x \in p^{-1}(\{x\})$ define $t_{X,E}(\lambda)(x)(v_x) = \lambda(v)(x)$ where v is some section extending v_x. The proof now proceeds as in [51, p.118].

□

Given an ε-hermitian vector bundle (E, φ) over X we get an ε-hermitian form on $\Gamma(X, E)$ by sending sections s and s' to the function $x \mapsto \varphi_x(s(x), s'(x))$.

Corollary 2.25: *Let X be a compact Hausdorff space and let A be the hermitian ring $C_F(X)$ with the involution inherited from the involution on F. Then the global sections functor extends to an equivalence of categories $\Gamma(X-) : {}_\varepsilon\mathscr{H}_{(,\tau)}F(X) \xrightarrow{\cong} {}_\varepsilon\mathscr{Q}(A, \tau)$.*

Proof. We show that the functor is fully faithful and essentially surjective. The functor $\Gamma(X,-)$ is essentially surjective because, by Theorem 2.23 and Lemma 2.24, every isomorphism from $\Gamma(X,E)$ to $\Gamma(X,E)^*$ must come from an isomorphism between the bundles E and E^*, so every form on $\Gamma(X,E)$ must come from one on E. The functor is fully faithful because the functor on the underlying category of vector bundles is so. $\square$

3 Classical constructions in algebraic K-theory

3.1 The functor K_0

Let $\mathscr{C}$ be a category equipped with a coherently associative and commutative bifunctor $\oplus : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ (see [34, ch.VII] for a definition). Assume also that the isomorphism classes of objects of $\mathscr{C}$ form a set, denote it by $[obj\mathscr{C}]$. Then the operation $[C]+[C'] = [C \oplus C']$ defines a semigroup structure on $[obj\mathscr{C}]$.

Definition 3.1. With $\mathscr{C}$ as above, the Grothendieck group $K_0(\mathscr{C})$ of $\mathscr{C}$ is the group completion of the semigroup of isomorphism classes of objects of $\mathscr{C}$.

The Grothendieck group of $\mathscr{C}$ has a universal property:
Given any additive homomorphism $f : [obj\mathscr{C}] \to G$ where G is an abelian group, there is a unique group homomorphism $\tilde{f}$ making the following diagram commute:

$$\begin{tikzcd}
{[obj\mathscr{C}]} \arrow[rr, "f"] \arrow[dr] & & G \\
& K_0(\mathscr{C}) \arrow[ur, "\tilde{f}"] &
\end{tikzcd}$$

Let A be a ring and $\mathscr{C} = \mathscr{P}(A)$ with direct sum of A-modules as the sum-operation. Then $K_0(A) = K_0(\mathscr{C})$ is the 0-th algebraic K-group of A, as defined in [38].

Example 3.2. Every vector space admits a basis, so a finitely generated projective module over a field F is determined up to isomorphism by its vector space dimension. In other words the map $\dim : [obj\mathscr{P}(F)] \to \mathbb{N}$ taking an isomorphism class to its dimension is an isomorphism. The map preserves addition, so by the universal property it extends to an isomorphism $K_0(F) \cong \mathbb{Z}$.

Example 3.3. Let D be a Dedekind domain. By a theorem of Steinitz [38, ch. 1] every module $M \in obj(\mathscr{P}(D))$ is isomorphic to one of the form $D^n \oplus I$ where I is a fractional ideal of D. The number n is uniquely determined and I is unique up to isomorphism. Furthermore, if $M_1 \cong D^m \oplus I_1$ and $M_2 \cong D^n \oplus I_2$ then there is an isomorphism $M_1 \oplus M_2 \cong D^{m+n} \oplus I_1 I_2$. It follows that there is a group isomorphism $K_0(D) \cong \mathbb{Z} \oplus \mathrm{Pic}(D)$.

Let (A,τ) be a hermitian ring, and $\mathscr{C} = {}_\varepsilon\mathscr{Q}(A,\tau)$ with $\perp$ as sum-operation. The group $K_0(\mathscr{C})$ is then the zeroth ε-hermitian K-group of (A,τ), denoted by ${}_\varepsilon KQ_0(A,\tau)$. As before the symbol τ

is often dropped from the notation. Recall the forgetful functor $U : iso({}_\varepsilon\mathscr{Q}(A,\tau)) \to iso(\mathscr{P}(A))$ and hyperbolic functor ${}_\varepsilon H : iso(\mathscr{P}(A)) \to iso({}_\varepsilon\mathscr{Q}(A,\tau))$ of Definition 2.13. Each of these functors preserves the zero objects, and there are isomorphisms

$$U\big((P,\varphi) \perp (Q,\psi)\big) \cong U\big((P,\varphi)\big) \oplus U\big((Q,\psi)\big)$$

and

$${}_\varepsilon H(P \oplus Q) \cong {}_\varepsilon H(P) \perp {}_\varepsilon H(Q).$$

Hence the functors ${}_\varepsilon H$ and U induce group homomorphisms $U_* : {}_\varepsilon KQ_0(A,\tau)) \to K_0(A)$ and ${}_\varepsilon H_* : K_0(A) \to {}_\varepsilon KQ_0(A,\tau))$. When the ring A is commutative the operation $[M]\cdot[N] = [M \otimes_A N]$ defines a commutative multiplication on $K_0(A)$, turning it into a ring with unit $[A]$. Similarly the tensor product of 1-hermitian modules turns ${}_1KQ_0(A, id_A)$ into a commutative ring with unit $[\langle 1\rangle_A]$ when A is commutative. Both U_* and ${}_\varepsilon H_*$ preserve multiplication, and U_* is a ring homomorphism. However, $U_*([{}_\varepsilon H(A)]) = [A \oplus A]$ while $U_*([\langle 1\rangle_A]) = [A]$ and since A is commutative $[A \oplus A] \neq [A]$[1]. Hence ${}_\varepsilon H_*$ does not preserve units and is not a ring homomorphism.

Since tensoring preserves projectives, a ring homomorphism $f : A \to B$ induces a functor $- \otimes_A B : \mathscr{P}(A) \to \mathscr{P}(B)$. By the elementary properties of the tensor product, the functor commutes with direct sums. This means that we get an additive map $[obj\mathscr{P}(A)] \to [obj\mathscr{P}(B)] \to K_0(B)$, and hence by the universal property of $K_0(-)$ a unique induced map $f_* : K_0(A) \to K_0(B)$. The same reasoning applies for ${}_\varepsilon K\mathscr{Q}(-)$ and hermitian rings and maps. The homomorphisms U_* and ${}_\varepsilon H_*$ are natural in the rings. A consequence is that we can use the K-groups (hermitian or otherwise) to compare rings. Isomorphic rings will have isomorphic K-groups. As a ring-theoretic invariant the functor ${}_\varepsilon KQ_0(-)$ is often more refined than $K_0(-)$, even in the simplest case when the involution on the ring is trivial and $\varepsilon = 1$.

Let S be a set. The free abelian monoid on S is denoted by $\mathbb{N}\{S\}$ and the free abelian group on S is denoted by $\mathbb{Z}\{S\}$.

Example 3.4. Consider the rings $\mathbb{R}$ and $\mathbb{C}$ with trivial involutions. The inclusion map $\mathbb{R} \hookrightarrow \mathbb{C}$ induces an isomorphism $K_0(\mathbb{R}) \to K_0(\mathbb{C})$, so K_0 is too "coarse" to tell the two fields apart. The situation for ${}_1KQ_0(-)$ is quite different. A non-degenerate 1-hermitian module over $\mathbb{R}$ with trivial involution is just a real vector space with a non-degenerate symmetric bilinear form. By Sylvester's theorem [32, p.34] all such vector spaces are up to isomorphism of the form

$$\underbrace{\langle 1\rangle_\mathbb{R} \perp \cdots \perp \langle 1\rangle_\mathbb{R}}_{m} \perp \underbrace{\langle -1\rangle_\mathbb{R} \perp \cdots \perp \langle -1\rangle_\mathbb{R}}_{n},$$

where m and n are uniquely determined by the space and the form. The monoid $[obj_1\mathscr{Q}(\mathbb{R}, id)]$ is therefore isomorphic to $\mathbb{N}\{\langle 1\rangle\} \times \mathbb{N}\{\langle -1\rangle\}$, so we get an isomorphism ${}_1KQ_0(\mathbb{R}) \cong \mathbb{Z}\{\langle 1\rangle\} \oplus$

[1]Choose a prime ideal $\mathfrak{p}$ in A. The function $\mathrm{rk}_\mathfrak{p} : [obj\mathscr{P}(A)] \to \mathbb{Z}$ sending $[M]$ to the rank of $M \otimes A_\mathfrak{p}$ as an $A_\mathfrak{p}$-module is both additive and multiplicative, hence it factors through $K_0(A)$. Clearly $\mathrm{rk}_\mathfrak{p}(A \oplus A) = 2$ and $\mathrm{rk}_\mathfrak{p}(A) = 1$.

$\mathbb{Z}\{\langle -1\rangle\}$. The multiplication is in both cases given by $(m,n)\cdot(k,l) = (km+nl, ml+nk)$. Any complex vector space V equipped with a symmetric bilinear form is isomorphic to an orthogonal sum of $\dim V$ copies of $\langle 1\rangle_{\mathbb{C}}$ [32, ch.I]. It follows that there is a ring isomorphism ${}_1KQ_0(\mathbb{C}) \cong \mathbb{Z}$. The map induced by the inclusion of $\mathbb{R}$ into $\mathbb{C}$ sends both $[\langle 1\rangle_{\mathbb{R}}]$ and $[\langle -1\rangle_{\mathbb{R}}]$ to $[\langle 1\rangle_{\mathbb{C}}]$.

In general it is quite difficult to compute the groups ${}_\varepsilon KQ_0(A,\tau)$, but in the commutative case one has some general results. For commutative ring it makes sense to call ${}_1KQ_0(A, id)$ the 0-th *orthogonal* K-group $KO_0(A)$ of A and ${}_{-1}KQ_0(A, id)$ the 0-th *symplectic* K-group $KSp_0(A)$ of A. These names and notations sometimes appear in the older literature but for simplicity and compatibility with the modern literature we have chosen to stick with "ε–hermitian".

Lemma 3.5: *Let A a commutative local ring with $2 \in A^\times$. Then the group ${}_1KQ_0(A, id)$ is additively generated by the set $A^\times/(A^\times)^2$, called the unit square classes of A.*

Proof. This follows immediately from Lemma 2.19. □

Lemma 3.6: *Let A be a commutative ring which is local or a Dedekind domain. Then the group ${}_{-1}KQ_0(A, id)$ is isomorphic to the free group $\mathbb{Z}\{[{}_{-1}H(A)]\}$.*

Proof. This follows immediately from Lemma 2.20. □

Example 3.7. Over the ring $A = \mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ any finitely generated projective module is free. So, endowing A with the trivial involution, any 1-hermitian module (M,φ) is isomorphic to one of the form (A^n, P) where P is a symmetric matrix. If Q is any invertible $n\times n$-matrix, then multiplication by Q determines an isomorphism $(A^n, Q^T PQ) \xrightarrow{\cong} (A^n, P)$. There is a function

$$\mathrm{disc} : {}_1KQ_0(A) \to A^\times/(A^\times)^2$$

called the discriminant map, which sends a class $[(M,\varphi)]$ to the square class of the determinant of a representing matrix for φ. This map is well defined because

$$[\det(Q^T PQ)] = [\det(Q)^2 \det(P)] = [\det(P)].$$

It is a group homomorphism because the determinant distributes over block sums of matrices. The group of square classes of units of A is isomorphic to $\mathbb{Z}/2\{\sqrt{-1}\} \oplus \mathbb{Z}/2\{1+i\}$ and the map $A^\times/(A^\times)^2 \to {}_1KQ_0(A)$ given by $[\sqrt{-1}] \mapsto [\langle\sqrt{-1}\rangle_A] - [\langle 1\rangle_A]$ and $[1+\sqrt{-1}] \mapsto [\langle 1+\sqrt{-1}\rangle_A - [\langle 1\rangle_A]$ is a section of the discriminant map. The rank map $\mathrm{rk} : {}_1K\mathscr{Q}_0(A) \to \mathbb{Z}$ is clearly surjective and has a section given by $1 \mapsto \langle 1\rangle_A$. It follows that there is an isomorphism

$${}_1KQ_0(A) \cong \mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus G$$

where G is some abelian group.

The cokernel of the hyperbolic map ${}_1H : K_0(A) \to {}_1KQ_0(A)$ is the classical Witt ring of A as defined in [39]. We know that A is a Dedekind domain and that its class group is trivial,

so by Example 3.3 there is an isomorphism $K_0(A) \cong \mathbb{Z}$. The element -1 of A is a square, so there is an isomorphism between ${}_1H(A) \cong \langle 1\rangle_A \perp \langle -1\rangle$ and $\langle 1\rangle \perp \langle 1\rangle$. This means that $K_0(A)$ is mapped onto $2\mathbb{Z}$ in the free summand generated by $[\langle 1\rangle]$ and that the cokernel of ${}_1H$ is isomorphic to $\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus G$. However, from [14] we know that the cokernel is isomorphic to $\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$ so we must have $G = 0$.
For $\varepsilon = -1$ Lemma 3.6 gives an isomorphism

$$ {}_{-1}KQ_0(A) \cong \mathbb{Z}. $$

Example 3.8. Let p be a prime different from 2. The set of square classes $\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$ has two elements, represented by 1 and s in $\mathbb{F}_p$. By Lemma 3.5 ${}_1KQ_0(\mathbb{F}_p)$ in generated by square classes, so this group can be generated by the two elements $\langle 1\rangle$ and $\langle s\rangle$. The same considerations of $\dim$ and disc as in Example 3.7 show that ${}_1KQ_0(\mathbb{F}_p)$ has a free summand generated by $\langle 1\rangle$ and a $\mathbb{Z}/2$-summand generated by $\langle s\rangle$. This exhausts the possible generators of the group so we have an isomorphism

$$ {}_1KQ_0(\mathbb{F}_p) \cong \mathbb{Z}\oplus\mathbb{Z}/2. $$

When $\varepsilon = -1$ Lemma 3.6 gives an isomorphism

$$ {}_{-1}KQ_0(\mathbb{F}_p) \cong \mathbb{Z}. $$

3.2 K_0 in topology

Let X be a topological space and let $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ as before. The Whitney sum of vector bundles is coherently commutative and associative and defines a semi-group structure on $[\mathscr{V}_F(X)]$. The group completion of $[\mathscr{V}_F(X)]$ with respect to this operation is the group $K_F^0(X) = K_0(\mathscr{V}_F(X))$. Let $f : X \to Y$ be a continuous map, E and E' bundles over Y. Since there is a natural isomorphism $f^*(E\oplus E') \cong f^*(E)\oplus f^*(E')$, the map $f^* : [\mathscr{V}_F](Y) \to [\mathscr{V}_F](X)$ is additive and hence induces a map on group completions $K_F^0(Y) \to K_F^0(X)$ which we will also denote by f^*. It follows that $X \mapsto K_F^*(X)$ defines a homotopy invariant functor $Top \to Ab$. This functor enjoys many nice properties. When restricted to CW-complexes (paracompact Hausdorff suffices) it is representable up to homotopy. For pointed topological spaces X and Y, let $[X, Y]$ denote the (unpointed) homotopy classes of maps $X \to Y$. There is a natural isomorphism $[X, \mathbb{Z}\times BG_F] \cong K_F^0(X)$ where G_F is defined by

$$ G_{\mathbb{R}} = O = \bigcup_{n\in\mathbb{N}} O(n) $$
$$ G_{\mathbb{C}} = U = \bigcup_{n\in\mathbb{N}} U(n) $$
$$ G_{\mathbb{H}} = Sp = \bigcup_{n\in\mathbb{N}} Sp(n) $$

and BG_F is the classifying space of G_F [25, p. 109]. From Theorem 2.23 we have the following:

Theorem 3.9: *Let X be a compact Hausdorff space. The global section functor induces a natural isomorphism*

$$K_F^0(X) \xrightarrow{\cong} K_0(C_F(X))$$

where F denotes $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

The same constructions can be carried out with ε-hermitian F-vector bundles, giving ε-hermitian topological K-theory which is also a functor $KH_{(F,\tau)} : HoTop \to Ab$. Here there are several possible choices for F and how to make it a hermitian ring, each choice giving a distinct functor. However, every choice gives a functor expressible in an elementary way using the functors $K_{\mathbb{R}}$, $K_{\mathbb{C}}$ and $K_{\mathbb{H}}$. When $\mathbb{R}$ and $\mathbb{C}$ are given the trivial involutions, one has the following isomorphisms:

$$\begin{aligned}
{}_1KH^0_{\mathbb{C}}(-) &\cong K^0_{\mathbb{R}}(-)\\
{}_{-1}KH^0_{\mathbb{C}}(-) &\cong K^0_{\mathbb{H}}(-)\\
{}_1KH^0_{\mathbb{R}}(-) &\cong K^0_{\mathbb{R}}(-) \times K^0_{\mathbb{R}}(-)\\
{}_{-1}KH^0_{\mathbb{R}}(-) &\cong K^0_{\mathbb{C}}(-)
\end{aligned}$$

From Theorem 2.25 we have the following:

Theorem 3.10: *Let X be a compact Hausdorff space and let (F, τ) be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ with the continuous involution τ. The global section functor induces a natural isomorphism*

$${}_\varepsilon KH^0_{(F,\tau)}(X) \xrightarrow{\cong} {}_\varepsilon KQ_0(C_{(F,\tau)}(X)).$$

for ε equal to 1 or -1.

Every topological space X comes with a map to the one point space $*$. This induces a map $K_F^0(\{*\}) \to K_F^0(X)$ whose cokernel is denoted by $\tilde{K}_F^0(X)$ and is called the reduced K_F-theory group of X. For $n \geq 1$ we define the $-n$-th K_F-group of X by $K_F^{-n}(X) = \tilde{K}_F^0(\Sigma^n(X_+))$, where Σ^n is the n-th reduced suspension functor and X_+ is the space X with a disjoint basepoint added. Write $[X, Y]_*$ for the based homotopy classes of maps $X \to Y$. For $n \geq 1$ there are natural isomorphisms

$$K_F^{-n}(X) \cong [X, \Omega^n BG_F]_* \cong \pi_n(BG_F^X).$$

The celebrated Bott periodicity theorem [30] states that $BG_F \simeq \Omega^{k_F} BG_F$ where $k_{\mathbb{C}} = 2$ and $k_{\mathbb{R}} = k_{\mathbb{H}} = 8$. This means that $K_F^{-n}(X) \cong K_F^{-n+k_F}(X)$ and this is used to define topological K^n-functors for all integers n. This motivates the search for higher (and lower) K-groups in other settings, in particular higher algebraic and hermitian K-groups.

3.3 The functor K_1

Let $\mathscr{C}$ be a category with sum operation as above. Define $aut(\mathscr{C})$ to be the category whose objects are pairs (M, α) where M is an object of $\mathscr{C}$ and $\alpha : M \to M$ is an isomorphism. A morphism

$(M,\alpha) \to (N,\beta)$ is a morphism $f : M \to N$ in $\mathscr{C}$ such that $\beta \circ f = f \circ \alpha$. Define the Bass group $K_1(\mathscr{C})$ to be the abelian group generated by the isomorphism classes $[(M,\alpha)]$, subject to the relations

$$\begin{aligned}
[(M,\alpha)] + [(N,\beta)] =& [(M \oplus N, \alpha \oplus \beta)] \\
[(M,\alpha)] + [(M,\alpha')] =& [(M, \alpha \circ \alpha').]
\end{aligned}$$

When $\mathscr{C} = \mathscr{P}(A)$ the group $K_1(\mathscr{C})$ can be described entirely in terms of general linear groups over A. There is a sequence of inclusions

$$GL_1(A) \hookrightarrow GL_2(A) \hookrightarrow \cdots GL_n(A) \hookrightarrow GL_{n+1}(A) \hookrightarrow \cdots$$

given at each stage by

$$P \mapsto \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}.$$

The colimit of the sequence is denoted by $GL(A)$.

Let $e_{i,j}(a)$ denote the matrix with a in the (i,j) position $(i \neq j)$, 1's on the diagonal and zeros everywhere else. Such matrices are called elementary since they represent the elementary row and column operations. They generate the subgroup $E_n(A)$ of $GL_n(A)$ called the n-th elementary group of A. In the sequence above $E_n(A)$ maps to $E_{n+1}(A)$ and the colimit of the subgroups is denoted by $E(A)$.

The following lemma is well known, and a nice proof can be found in [38, p.25].

Lemma 3.11: *(Whitehead) For any ring A the group of elementary matrices $E(A)$ is the commutator subgroup of $GL(A)$.*

Furthermore the equality $e_{ij}(a) = [e_{ik}(a), e_{kj}(1)]$ when i, j and k are distinct [38, p. 39] implies that $E(A)$ equals its commutator subgroup.

Definition 3.12. The first algebraic K-group of A is defined by $K_1(A) = GL(A)/E(A)$, or equivalently $K_1(A) = GL(A)_{ab}$, the abelianization of $GL(A)$.

The construction is clearly functorial and defines a functor $K_1 : Ring \to Ab$ from rings to abelian groups.

Proposition 3.13: *There is a natural isomorphism $K_1(\mathscr{P}(A)) \cong K_1(A)$.*

Proof. Any finitely generated projective module can be embedded as a direct summand of a free module of finite rank. This means that the subcategory of finitely generated free modules $\mathscr{F}(A)$ of $\mathscr{P}(A)$ is full and cofinal in the sense of [4, I,8.3]. Then by [4, VII,2.2] the map $K_1(\mathscr{F}(A)) \to K_1(\mathscr{P}(A))$ is an isomorphism. The group $K_1(\mathscr{F}(A))$ is isomorphic to $K_1(A)$ by [4, IX.1.2]. $\square$

Example 3.14. In the commutative case some of the structure of $K_1(A)$ is known. The determinant $\det : GL(A) \to A^*$ induces a map $K_1(A) \to A^*$ which is split by the inclusion of A^* into $GL(A)$ as $GL_1(A)$. When A is local or euclidean it is quite easy to show that the map $K_1(A) \to A^*$ is an isomorphism (see e.g. [46, Ch.2]). For rings of integers in number fields the result is also true, but the proof is much more difficult. It was shown by Bass, Milnor and Serre in [6].

We now examine how these ideas turn out in the ε-hermitian setting. For simplicity we assume that 2 is a unit in any ring we consider. This greatly simplifies the statements of the definitions and theorems to come but it is only essential for proving that the different approaches to KQ_1 are equivalent. For a completely general treatment the reader is again referred to [3]. As we saw in lemma 2.17, 2 being invertible implies that any ε-hermitian module (M, φ) is an orthogonal summand of ${}_\varepsilon H(A^n)$ for some n. This again implies that $O(M, \varphi)$ is a subgroup of ${}_\varepsilon O_{n,n}(A, \tau)$. There is a sequence of inclusions

$$ {}_\varepsilon O_{1,1}(A,\tau) \hookrightarrow {}_\varepsilon O_{2,2}(A,\tau) \hookrightarrow \cdots {}_\varepsilon O_{n,n}(A,\tau) \hookrightarrow {}_\varepsilon O_{n+1,n+1}(A,\tau) \hookrightarrow \cdots \qquad (*) $$

given at each stage by

$$ \begin{pmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{pmatrix} \mapsto \begin{pmatrix} P_{1,1} & 0 & P_{1,2} & 0 \\ 0 & 1 & 0 & 1 \\ P_{2,1} & 0 & P_{2,2} & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} $$

where the $P_{i,j}$s are $n \times n$-matrices. The colimit of the sequential diagram (*) defines the infinite ε-orthogonal group ${}_\varepsilon O(A, \tau)$.

Definition 3.15. Let (A, τ) be a hermitian ring with 2 a unit in A. The first ε-hermitian K-group of (A, τ) is ${}_\varepsilon KQ_1(A, \tau) = {}_\varepsilon O(A, \tau)_{ab}$, the abelianization of ${}_\varepsilon O(A, \tau)$.

Proposition 3.16: *Let (A, τ) be a hermitian ring in which 2 is a unit. Then there is a natural isomorphism ${}_\varepsilon KQ_1(A, \tau) \cong K_1({}_\varepsilon\mathscr{Q}(A, \tau))$.*

Proof. By lemma 2.17 any ε-hermitian module can be embedded as an orthogonal summand of a hyperbolic space, because 2 is a unit in A. The result now follows by the same argument as in the proof of Proposition 3.13. □

Example 3.17. Let (A, id_A) be a commutative hermitian ring in which 2 is a unit. If in addition A is euclidean then the group ${}_{-1}O(A) = Sp(A)$ (the symplectic group of A) is perfect [26]. It follows that the group ${}_{-1}KQ_1(A) = 0$ for such a ring. Examples include the rings $\mathbb{Z}[\frac{1}{2}]$, $\mathbb{Z}[\sqrt{-1}, \frac{1}{2}]$ and $k[x]$ where k is a field of characteristic different from 2.

Example 3.18. For finite fields of odd order the hermitian K-groups were computed in [19]. In degree 1 they are ${}_{-1}KQ_1(\mathbb{F}_q) = 0$ and ${}_1KQ_1(\mathbb{F}_q) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Example 3.19. Let $A = \mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$. Then from [5, 4.7.6] we have an isomorphism ${}_{-1}KQ_1(A) \cong A^\times/(A^\times)^2 \oplus \mathbb{Z}/2$. By Example 1.16 there is a n isomorphism $A^\times/(A^\times)^2 \cong (\mathbb{Z}/2)^2$ which gives ${}_{-1}KQ_1(A) \cong (\mathbb{Z}/2)^3$. The result [5, 4.7.6] also applies to $\mathbb{F}_5$ and the map $A \twoheadrightarrow A/(1+\sqrt{-1}) \xrightarrow{\cong} \mathbb{F}_5$ induces a commutative square

$$\begin{tikzcd}
{}_{-1}KQ_1(A) \arrow[r, "\cong"] \arrow[d] & A^\times/(A^\times)^2 \oplus \mathbb{Z}/2 \arrow[d] \\
{}_{-1}KQ_1(\mathbb{F}_5) \arrow[r, "\cong"] & \mathbb{F}_5^\times/(\mathbb{F}_5^\times)^2 \oplus \mathbb{Z}/2.
\end{tikzcd}$$

The right hand vertical map sends the classes of $\sqrt{-1}$ and $1+\sqrt{-1}$ to the generator of $\mathbb{F}_5^\times/(\mathbb{F}_5^\times)^2$ and restricts to an isomorphism on the $\mathbb{Z}/2$-summand.

The elementary matrices have ε-hermitian analogues. Since this is less known than in the general linear case above we give a detailed treatment. To simplify the notation, set

$$\alpha_{ij}^{\varepsilon}(a) = \begin{pmatrix} 0 & & a_{ij} \\ & \ddots & \\ -\varepsilon\bar{a}_{ji} & & 0 \end{pmatrix} \in M_{n,n}(A)$$

where a_{ij} means an a in the (i,j)-spot for $i \neq j$. Further, let $\alpha_{ii}^{\varepsilon}(a)$ be an $n \times n$-matrix with an a such that $a = -\varepsilon\bar{a}$ in the (i,i)-spot and zeros everywhere else. The n-th ε-elementary group ${}_{\varepsilon}E_{n,n}(A,\tau) \subseteq {}_{\varepsilon}O_{n,n}(A,\tau)$ of (A,τ) is generated by matrices of the forms

$$H(e_{ij}(a)) = \begin{pmatrix} e_{ij}(a) & 0 \\ 0 & e_{ji}(-\bar{a}) \end{pmatrix}$$

$$E_{i,n+j}(a) = \begin{pmatrix} I & \alpha_{ij}^{\varepsilon}(a) \\ 0 & I \end{pmatrix}$$

$$E_{n+i,j}(a) = \begin{pmatrix} I & 0 \\ \alpha_{ij}^{\bar{\varepsilon}}(a) & I \end{pmatrix}$$

$$E_{i,n+i}(a) = \begin{pmatrix} I & \alpha_{ii}^{\varepsilon}(a) \\ 0 & I \end{pmatrix}$$

$$E_{n+i,i}(a) = \begin{pmatrix} I & 0 \\ \alpha_{ii}^{\bar{\varepsilon}}(a) & I \end{pmatrix}$$

where $i \neq j$. A set of relations for these generators can be found in [3, p. 43].

The group ${}_{\varepsilon}E_{n,n}(A,\tau) \subseteq {}_{\varepsilon}O_{n,n}(A,\tau)$ lands in ${}_{\varepsilon}E_{n+1,n+1}(A,\tau)$ under the inclusion of ${}_{\varepsilon}O_{n,n}(A,\tau)$ into ${}_{\varepsilon}O_{n+1,n+1}(A,\tau)$. Set

$${}_{\varepsilon}E(A,\tau) = \operatorname*{colim}_n {}_{\varepsilon}E_{n,n}(A,\tau).$$

We now proceed to show that for $n \geq 3$ any generator is a product of commutators of other generators. The case of $H(e_{ij}(a))$ is particularly simple: Since $e_{ij}(a) = [e_{ik}(a), e_{kj}(1)]$, we have $H(e_{ij}(a)) = H([e_{ik}(a), e_{kj}(1)]) = [H(e_{ik}(a)), H(e_{kj}(1))]$ when i, j, k are distinct.

For the other generators, one has the following relations [3, p. 29]:

$$\begin{aligned} E_{i,n+k}(a) =& [E_{j,n+k}(a), H(e_{ij}(1))] \\ E_{n+j,k}(a) =& [E_{n+i,k}(a), H(e_{ij}(-1))] \\ E_{j,n+j}(a)E_{i,n+j}(a) =& [E_{j,n+j}(a), H(e_{ij}(1))] \\ E_{n+i,i}(a)E_{n+j,i}(a) =& [E_{n+i,i}(a), H(e_{ij}(-1))]. \end{aligned}$$

Hence we obtain the following lemma.

Lemma 3.20: *Let (A, τ) be a hermitian ring in which 2 is a unit. Then for $n \geq 3$ we have*

$$[{}_\varepsilon E_{2n,2n}(A, \tau), {}_\varepsilon E_{2n,2n}(A, \tau)] = {}_\varepsilon E_{2n,2n}(A, \tau)$$

An immediate consequence of Lemma 3.20 is that $[{}_\varepsilon E(A, \tau), {}_\varepsilon E(A, \tau)] = {}_\varepsilon E(A, \tau)$. The following lemma is checked by multiplying out both sides of the equality.

Lemma 3.21: *(Orthogonal Whitehead lemma) Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in {}_\varepsilon O_{n,n}(A, \tau)$. Then*

$$\begin{pmatrix} \alpha & & \beta & \\ & I & & \\ \gamma & & \delta & \\ & & & I \end{pmatrix}\begin{pmatrix} A & & B & \\ & I & & \\ C & & D & \\ & & & I \end{pmatrix} = \begin{pmatrix} I & & & -\beta \\ \bar{\delta} & I & \varepsilon\bar{\beta} & \\ & & I & -\delta \\ & & & I \end{pmatrix}\begin{pmatrix} & I & & \\ -I & & & \\ & & & I \\ & & -I & \end{pmatrix}\begin{pmatrix} A & & B & \\ & \alpha & & \beta \\ C & & D & \\ & \gamma & & \delta \end{pmatrix}\begin{pmatrix} I & & & -\varepsilon\bar{B} \\ A & I & B & \\ & & I & -\bar{A} \\ & & & I \end{pmatrix}\begin{pmatrix} I & -\bar{D} & & \\ & I & & \\ & -\bar{\varepsilon}\bar{C} & I & \\ C & & D & I \end{pmatrix}.$$

On the right hand side of the equality, all the matrices except the one in the middle are automatically in ${}_\varepsilon E_{2n,2n}(A, \tau)$.

This leads to the following crucial result:

Theorem 3.22: *For a hermitian ring (A, τ) in which 2 is a unit, the commutator subgroup of ${}_\varepsilon O(A, \tau)$ equals the group ${}_\varepsilon E(A, \tau)$ of ε-elementary matrices.*

Proof. For matrices $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $Q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ define

$$P \perp Q = \begin{pmatrix} A & & B & \\ & \alpha & & \beta \\ C & & D & \\ & \gamma & & \delta \end{pmatrix}.$$

Let $P \in {}_\varepsilon O_{n,n}(A, \tau)$. Then $P \perp I$ coincides with the image of P in ${}_\varepsilon O_{2n,2n}(A, \tau)$ under the usual inclusion. Observe that $(P \perp I)^{-1}(I \perp P) = P^{-1} \perp P$, and by Lemma 3.21 applied to the matrices P and P^{-1}, the matrix $E = P^{-1} \perp P$ is in ${}_\varepsilon E_{2n,2n}(A, \tau)$. For any $Q \in {}_\varepsilon O_{n,n}(A, \tau)$,

$$(Q \perp I)(P \perp I) = (Q \perp I)(I \perp P)E = (Q \perp P)E.$$

By the same argument as above $E' = (Q \perp I)^{-1}(I \perp Q)$ is also in ${}_\varepsilon E_{2n,2n}(A,\tau)$, which gives

$$(Q \perp P)E = (I \perp PQ)E'E = (PQ \perp I)E''E'E = (P \perp I)(Q \perp I)E''E'E,$$

where $E'' = Q^{-1}P^{-1} \perp PQ \in {}_\varepsilon E_{2n,2n}(A,\tau)$. It follows that $(Q^{-1}P^{-1}QP) \perp I = E''E'E \in {}_\varepsilon E_{2n,2n}(A,\tau)$. □

Corollary 3.23: *The group ${}_\varepsilon K_1\mathcal{Q}(A,\tau)$ is naturally isomorphic to ${}_\varepsilon O(A,\tau)/{}_\varepsilon E(A,\tau)$, since this is the abelianization of ${}_\varepsilon O(A,\tau)$.*

3.4 The functor K_2

Definition 3.24. A central extension of a group G is a pair (E,ψ) where E is a group and $\psi : E \to G$ is a surjective group homomorphism such that $\ker(\psi)$ is a central subgroup of E. A morphism from (E,ψ) to (E',ψ') of central extensions of G is a homomorphism $E \to E'$ over G.

A central extension (U,υ) of G is called universal if it is initial among the central extensions of G.

Definition 3.25. A group G is called perfect if $G = [G,G]$.

For a perfect group we have $G_{ab} = 0$, hence the first group homology group (see [56, Ch.6]) $H_1^{grp}(G;\mathbb{Z})$, which is isomorphic to G_{ab}, vanishes.

Proposition 3.26: *A group G admits a universal central extension if and only if it is perfect.*

Proof. [38, p.45] □

The kernel of the universal central extension can be described in more familiar terms:

Proposition 3.27: *Let G be a perfect group and (U,υ) be its universal central extension. There is a natural isomorphism* $\ker(\upsilon) \cong H_2^{grp}(G;\mathbb{Z})$.

Above we had two examples of (types of) perfect groups $E(A)$ and ${}_\varepsilon E(A,\tau)$. In these cases there is a convenient construction of the universal central extensions called Steinberg groups. The Steinberg group $St(A)$ of A is generated by symbols $x_{i,j}^a$ subject to relations [38, p.40] which only derive from the "shape" of the matrices $e_{i,j}(a)$ and the additive and multiplicative relations among the elements of A. Let $\varphi_A : St(A) \to E(A)$ be given by $x_{i,j}^a \mapsto e_{i,j}(a)$, then $(St(A),\varphi_A)$ is a universal central extension of $E(A)$. We define the second algebraic K-group $K_2(A)$ of A to be the kernel of this map. Thus we get a short exact sequence of groups

$$1 \longrightarrow K_2(A) \longrightarrow St(A) \xrightarrow{\varphi_A} E(A) \longrightarrow 1,$$

which is natural in A. By Proposition 3.27 there is a natural isomorphism $K_2(A) \cong H_2(E(A);\mathbb{Z})$. The construction for ${}_\varepsilon E(A,\tau)$ is analogous but slightly more complicated since the set of generators is more complicated (see [3, p.43]). There are generators $r_{ij}(a)$, $l_{ij}(a)$ and $H_{ij}(a)$ which map to $E_{i,n+j}(a)$, $E_{n+i,j}(a)$ and $H(e_{ij}(a))$, respectively, in ${}_\varepsilon E(A,\tau)$. The group ${}_\varepsilon KQ_2(A,\tau)$ is defined to be the kernel of the surjection ${}_\varepsilon St(A,\tau) \twoheadrightarrow {}_\varepsilon E(A,\tau)$.

Example 3.28. There is an isomorphism $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$, see [38, Ch. 10] for a computation.

Example 3.29. While K_2 of a finite field always vanishes (see [38]) we have ${}_1KQ_2(\mathbb{F}_q) \cong \mathbb{Z}/2$ and ${}_{-1}KQ_2(\mathbb{F}_q) = 0$ when q is odd [19].

Example 3.30. In [10] Berrick and Karoubi show that ${}_1KQ_2(\mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}/2\oplus\mathbb{Z}/2$ and ${}_{-1}KQ_2(\mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}$.

4 The Quillen plus-construction and higher K-theory

With the definitions from the previous chapter it is not clear how the different K-functors relate to each other, or how one would define K_n for $n \geq 3$. The plus construction gives a partial answer to the first question and a complete answer to the second.

There are several approaches to higher K-theory, all of them giving isomorphic K-groups. Of these the most intrinsic and general is the Waldhausen $S_\bullet$-construction, this can be applied both to algebraic K-theory and to hermitian K-theory (see e.g. [49, 2.10]). Although the $S_\bullet$-construction is more conceptually appealing than the plus construction, the plus construction has the advantage that it is much easier to define and, for our purposes, also easier to work with.

4.1 The plus-construction

Recall that for any group G there is a space BG called the classifying space of G. It satisfies the following properties (see [35, 16.5])

- BG is a $K(G,1)$, i.e. $\pi_1(BG) \cong G$ and all the other homotopy groups of BG are trivial.
- BG is unique up to homotopy equivalence.
- BG is functorial in G.
- For each n there is a natural isomorphism $H_n(BG;\mathbb{Z}) \cong H_n^{grp}(G;\mathbb{Z})$.

From this and the isomorphisms $H_1^{grp}(GL(A);\mathbb{Z}) \cong K_1(A)$ and $H_1^{grp}(E(A);\mathbb{Z}) \cong K_2(A)$ we see that the first and second K-groups of a ring A can be described as homology groups of the space $BGL(A)$. The same holds for ε-hermitian K-theory and hermitian rings. It seems reasonable that the definition of higher K-groups should involve the spaces $BGL_n(A)$ and $B_\varepsilon O(A,\tau)$. This idea is formalized by using the Quillen plus-construction.

Theorem 4.1: *Let X be a pointed CW-complex and P a perfect normal subgroup of $\pi_1(X)$. There exists a space X_P^+ and a map $f : X \to X_P^+$ with the following properties:*

1. *The group P is the kernel of the map $\pi_1(f) : \pi_1(X) \to \pi_1(X_P^+)$ and the induced map $\bar{f} : \pi_1(X)/P \to \pi_1(X_P^+)$ is an isomorphism.*

2. *The induced map on homology $H_*(f) : H_*(X) \to H_*(X_P^+)$ is an isomorphism.*

3. *The pair (X_P^+, f) is universal up to homotopy with respect to the above properties.*

Proof. We prove the theorem for connected X. When X is disconnected we can use map f for the basepoint component and the identity maps for the other components.

Choose a set of generators for P and for each generator map $a_i : S^1 \to X$ representing it. Now form the space Y by attaching discs D^2 to X along the maps a_i. This gives a diagram

$$\begin{tikzcd} \vee_i S^1 \arrow[r, "\vee_i a_i"] \arrow[d, tail] & X \arrow[d, tail] \\ \vee_i D^2 \arrow[r] & Y \end{tikzcd}$$

where the vertical maps are cofibrations. By the Van Kampen theorem the map $X \to Y$ satisfies condition 1 above. Let $\tilde{X}$ be the covering space of X with transformation group $\pi_1(X)/P$ so that $\pi_1(\tilde{X}) \cong P$. Each a_i lifts to a loop at the base point of $\tilde{X}$ and there are translates of these lifts by $\pi_1(X)/P$. Attaching discs D^2 along all the translates gives a covering space $p_Y : \tilde{Y} \to Y$. In fact, $\pi_1(\tilde{Y}) = 0$, so $\tilde{Y}$ is the universal covering space of Y. Since P is perfect, the maps a_i used to attach cells to $\tilde{X}$ to obtain $\tilde{Y}$ are homotopically products of commutators, hence the boundaries of these cells vanish in the cellular chain complex of $\tilde{Y}$. It follows that $H_2(\tilde{Y}) \cong H_2(\tilde{X}) \oplus F$ where F is a free $\mathbb{Z}[\pi_1(X)/P]$-module on the 2-cells attached to X to obtain Y. Since $\tilde{Y}$ is simply connected the Hurewicz map $\pi_2(\tilde{Y}) \to H_2(\tilde{Y})$ is an isomorphism. Choose maps $b_i : S^2 \to \tilde{Y}$ which map to a basis for F as a $\mathbb{Z}[\pi_1(X)/P]$-module. Attach 3-cells to Y along $p_Y \circ b_i$ to obtain the space X_P^+,

$$\begin{tikzcd} \vee_i S^2 \arrow[r, "\vee_i b_i"] \arrow[d, tail] & Y \arrow[d, tail] \\ \vee_i D^3 \arrow[r] & X_P^+ \end{tikzcd}$$

and construct a covering space $\tilde{X}_P^+$ by translating cells as above. Since attaching 3-cells does not affect fundamental groups, the map $X \to X_P^+$ also satisfies condition 1. The cellular chain complex $C_*(X^+, X)$ looks like

$$\cdots \longrightarrow 0 \longrightarrow F \xrightarrow{\partial_2} F \longrightarrow 0 \longrightarrow 0$$

where ∂_2 is an isomorphism by construction. Hence the map $X \to X_P^+$ induces a chain equivalence on cellular chains and an isomorphism on homology. For a proof of the universality condition see for instance [46, p.269]

□

Let G be a group whose commutator subgroup E is perfect. Then the inclusion map $E \hookrightarrow G$ induces a covering space $BE \to BG$ with transformation group G/E. Applying the plus-construction gives

$$\begin{tikzcd} BE \arrow[r, "p"] \arrow[d] & BG \arrow[d] \\ BE^+ \arrow[r, "p^+"] & BG^+ \end{tikzcd}$$

where BE^+ is the universal covering space of BG^+. By construction $\pi_1(BG^+) \cong G/E = G_{ab}$. Since BE^+ is a covering space the covering map p^+ induces an isomorphism $\pi_2(BE^+) \cong \pi_2(BG^+)$. The space BE^+ is simply connected, so the Hurewicz map $\pi_2(BE^+) \to H_2(BE^+;\mathbb{Z})$ is an isomorphism. By Theorem 4.1 the map $BE \to BE^+$ induces an isomorphism on homology and for any group K there is a natural isomorphism $H_2^{grp}(K;\mathbb{Z}) \cong H_2(BK,\mathbb{Z})$. It follows that there is an isomorphism $\pi_2(BG^+) \cong H_2^{grp}(E;\mathbb{Z})$.

These arguments apply to the cases $E(A) \subseteq GL(A)$ and ${}_\varepsilon E(A,\tau) \subseteq {}_\varepsilon O(A,\tau)$ yielding

$$\pi_1(BGL(A)^+) \cong GL(A)/E(A) = K_1(A)$$
$$\pi_2(BGL(A)^+) \cong H_2^{grp}(E(A);\mathbb{Z}) \cong K_2(A)$$

and

$$\pi_1(B_\varepsilon O(A,\tau)^+) \cong {}_\varepsilon O(A,\tau)/{}_\varepsilon E(A,\tau) = {}_\varepsilon KQ_1(A,\tau)$$
$$\pi_2(B_\varepsilon O(A,\tau)^+) \cong H_2^{grp}({}_\varepsilon E(A,\tau);\mathbb{Z}) \cong {}_\varepsilon KQ_2(A,\tau).$$

Looking at these isomorphisms it is tempting to take the homotopy groups of $BGL(A)^+$ and $B_\varepsilon O(A,\tau)^+$ as the definitions of higher K-groups and hermitian K-groups respectively, but this is problematic since there were so many choices involved in the proof of Theorem 4.1. The following discussion provides a solution to this problem.

Let $\mathscr{C}$ be the category of pairs (X, P_X) where X is a pointed CW-complex and P_X is a perfect normal subgroup of $\pi_1(X)$, and whose morphisms are continuous maps $f : X \to Y$ such that $f_*(P_X) \subseteq P_Y$. For an object (X,P) of $\mathscr{C}$, a map $X \to Y$ in Top_* satisfying the conditions of Theorem 4.1 is called a plus-construction on X with respect to P. The plus- construction defines a functor from $\mathscr{C}$ to $HoTop_*$ [46, p.268] and in general this cannot be lifted to a functor to Top_*. However, one is interested in having functors from rings and hermitian rings to pointed spaces taking A to $BGL(A)^+$ and (A,τ) to $B_\varepsilon O(A)^+$.

Lemma 4.2: *Let X be a pointed CW-complex and P a perfect normal subgroup of $\pi_1(X)$. Given a plus construction $f : X \to X_P^+$ and a map $g : X \to Y$ such that the normal closure of $f_*(P)$ in $\pi_1(Y)$ is the perfect subgroup $Q \subseteq \pi_1(Y)$, then the right hand vertical map in the pushout square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X_P^+ \arrow[r] & X_P^+ \cup_X Y \end{tikzcd}$$

is a plus construction on Y with respect to Q.

Proof. See [9, 4.21]. □

The initial object in the category of rings is $\mathbb{Z}$. Hence if we choose once and for all a plus construction for $BGL(\mathbb{Z})$ then for any ring A there is a pushout square

$$\begin{tikzcd} BGL(\mathbb{Z}) \arrow[r] \arrow[d] & BGL(A) \arrow[d] \\ BGL(\mathbb{Z})^+ \arrow[r] & BGL(\mathbb{Z})^+ \cup_{BGL(\mathbb{Z})} BGL(A). \end{tikzcd}$$

By [9, 9] the normal closure of $St(\mathbb{Z})$ in $St(A)$ is all of $St(A)$, hence the same is true for $E(\mathbb{Z})$ and $E(A)$. It follows from Lemma 4.2 above that the left hand vertical morphism is a plus construction on $BGL(A)$ with respect to $E(A)$. We set

$$BGL(A)^+ = BGL(\mathbb{Z})^+ \cup_{BGL(\mathbb{Z})} BGL(A)$$

and call it *the* plus-construction on $BGL(A)$. This defines a functor $BGL(-)^+ : Ring \to Top_*$ as desired.

The same idea works in the ε-hermitian case when 2 is a unit in the ring and $\varepsilon = 1$ or $\varepsilon = -1$. Then $(\mathbb{Z}[\frac{1}{2}], id)$ is the initial object and we fix once and for all a plus construction on $B_\varepsilon O(\mathbb{Z}[\frac{1}{2}], id) = B_\varepsilon O(\mathbb{Z}[\frac{1}{2}])$. As in the general linear case we show that the normal closure of ${}_\varepsilon St(\mathbb{Z}[\frac{1}{2}])$ is all of ${}_\varepsilon St(A, \tau)$. First we use the hyperbolic map to identify $St(A)$ with a subgroup of ${}_\varepsilon St(A, \tau)$ and then we use the ε-hermitian Steinberg-relations [3, 3.16] to express the generators of ${}_\varepsilon St(A, \tau)$ as products of commutators of elements in $St(A)$ and ${}_\varepsilon St(\mathbb{Z}[\frac{1}{2}])$. It is this last step that requires $\varepsilon \in \{-1, 1\}$ and $2 \in A^\times$.

Since 2 is a unit in A we have a sequence of inclusions

$$St(\mathbb{Z}) \hookrightarrow St(\mathbb{Z}[\tfrac{1}{2}]) \hookrightarrow St(A).$$

It is easy to see that the normal closure of $St(\mathbb{Z}[\frac{1}{2}])$ in $St(A)$ is all of $St(A)$, so again this is true for elementary groups. The hyperbolic map identifies $St(\mathbb{Z}[\frac{1}{2}])$ and $St(A)$ with subgroups of

${}_{\varepsilon}St(\mathbb{Z}[\frac{1}{2}])$ and ${}_{\varepsilon}St(A,\tau)$ respectively:

$$\begin{tikzcd}
St(\mathbb{Z}[\frac{1}{2}]) \arrow[r] \arrow[d, "{}_{\varepsilon}H"] & St(A) \arrow[d, "{}_{\varepsilon}H"] \\
{}_{\varepsilon}St(\mathbb{Z}[\frac{1}{2}]) \arrow[r] & {}_{\varepsilon}St(A,\tau).
\end{tikzcd}$$

Any normal subgroup of ${}_{\varepsilon}St(A,\tau)$ containing $St(\mathbb{Z}[\frac{1}{2}])$ must also contain $St(A)$. This applies to the normal closure N of ${}_{\varepsilon}St(\mathbb{Z}[\frac{1}{2}])$, so N must contain all generators of the form $H_{ij}(a)$ where $a \in A$ and $i \neq j$. From the ε-hermitian Steinberg-relation L5a) [3, 3.16] we get

$$[l_{ij}(1), H_{jk}(a)] = l_{ik}(a)$$

for i, j, k distinct, so $l_{ik}(a) \in N$. Similarly, by R5a)

$$[r_{ij}(1), H_{jk}(\bar{a})] = r_{ik}(a)$$

so the latter term is also in N. For $\varepsilon = 1$ this takes care of all the generators, so $N = {}_{1}St(A,\tau)$. When $\varepsilon = -1$ there are also the generators $r_{ii}(a)$ and $l_{jj}(a)$ where $a = \bar{a}$. using R5b) we get

$$[r_{ij}(a), H_{ij}(-\tfrac{1}{2})] = r_{ii}(a)$$

and L5b) gives

$$[l_{ij}(a), H_{ji}(-\tfrac{1}{2})] = l_{ii}(a).$$

As above this means that the normal closure of ${}_{\varepsilon}E(\mathbb{Z}[\frac{1}{2}])$ in ${}_{\varepsilon}E(A,\tau)$ is all of ${}_{\varepsilon}E(A,\tau)$ and hence that the right hand vertical map in the pushout square

$$\begin{tikzcd}
B_{\varepsilon}O(\mathbb{Z}[\frac{1}{2}]) \arrow[r] \arrow[d] & B_{\varepsilon}O(A,\tau) \arrow[d] \\
B_{\varepsilon}O(\mathbb{Z}[\frac{1}{2}])^{+} \arrow[r] & B_{\varepsilon}O(\mathbb{Z}[\frac{1}{2}])^{+} \cup_{B_{\varepsilon}O(\mathbb{Z}[\frac{1}{2}])} B_{\varepsilon}O(A,\tau).
\end{tikzcd}$$

is a plus construction on $B_{\varepsilon}O(A,\tau)$. We define this to be *the* plus-construction on $B_{\varepsilon}O(A,\tau)$.

4.2 Higher K-theory

Definition 4.3. Let A be a ring. The algebraic K-theory space of A is defined by

$$K(A) = K_0(A) \times BGL(A)^{+}$$

where $K_0(A)$ is given the discrete topology. The n-th algebraic K-group is defined by

$$K_n(A) = \pi_n(K(A)).$$

In general not much is known about the higher K-groups of rings. Quillen has shown that the K-groups of rings of integers in number fields are finitely generated but even the groups $K_n(\mathbb{Z})$ are unknown for most n.

Example 4.4. Quillen showed in [44] that for $n \geq 1$ and $q = p^r$ for some prime p, we have isomorphisms $K_{2n}(\mathbb{F}_q) = 0$ and $K_{2n-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^n - 1)$ for n odd.

Definition 4.5. Let (A, τ) be a hermitian ring in which 2 is a unit and let ε be 1 or -1. The ε-hermitian K-theory space of (A, τ) is defined by

$$_{\varepsilon}KQ(A, \tau) = {_{\varepsilon}KQ_0(A, \tau)} \times B_{\varepsilon}O(A, \tau)^+$$

where $_{\varepsilon}KQ_0(A, \tau)$ is given the discrete topology. The n-th ε-hermitian K-group is defined by

$$_{\varepsilon}KQ_n(A, \tau) = \pi_n({_{\varepsilon}KQ(A, \tau)}).$$

For ease of notation we will write simply A for (A, τ) from now on.

The hyperbolic functor $_{\varepsilon}H : \mathscr{P}(A) \to {_{\varepsilon}\mathscr{Q}(A)}$ and the forgetful functor $_{\varepsilon}U : {_{\varepsilon}\mathscr{Q}(A)} \to \mathscr{P}(A)$ induce maps $_{\varepsilon}H_* : K(A) \to {_{\varepsilon}KQ(A)}$ and $_{\varepsilon}U_* : {_{\varepsilon}KQ(A)} \to K(A)$ respectively. The homotopy fiber of $_{\varepsilon}H_* : K(A) \to {_{\varepsilon}KQ(A)}$ is denoted by $_{\varepsilon}U(A)$ and the homotopy fiber of $_{\varepsilon}U_* : {_{\varepsilon}KQ(A)} \to K(A)$ is denoted by $_{\varepsilon}V(A)$. We define $_{\varepsilon}U_n(A) = \pi_n({_{\varepsilon}U(A)})$ and $_{\varepsilon}V_n(A) = \pi_n({_{\varepsilon}V(A)})$. There are long exact sequences

$$\cdots \longrightarrow {_{\varepsilon}U_n(A)} \longrightarrow K_n(A) \longrightarrow {_{\varepsilon}KQ_n(A)} \longrightarrow {_{\varepsilon}U_{n-1}(A)} \longrightarrow \cdots$$

and

$$\cdots \longrightarrow {_{\varepsilon}V_n(A)} \longrightarrow {_{\varepsilon}KQ_n(A)} \longrightarrow K_n(A) \longrightarrow {_{\varepsilon}V_{n-1}(A)} \longrightarrow \cdots$$

Theorem 4.6: *(Karoubi[27]) Let A be a hermitian ring with $2 \in A^{\times}$. Then there is a natural homotopy equivalence*

$$_{\varepsilon}V(A) \simeq \Omega_{-\varepsilon}U(A)$$

This is one of the main tools for computing hermitian K-groups, particularly when the algebraic K-groups are already known. The degree shift induced on the level of homotopy groups allows for induction techniques like the one used in [10] to compute the hermitian K-theory of $\mathbb{Z}[\frac{1}{2}]$.

Definition 4.7. The n-th Witt group of A is the cokernel of the hyperbolic map

$$_{\varepsilon}W_n(A) = coker(K_n(A) \to {_{\varepsilon}KQ_n(A)}).$$

The n-th co-Witt group is defined by

$$_{\varepsilon}W'_n(A) = ker({_{\varepsilon}KQ_n(A)} \to K_n(A)).$$

The group $\mathbb{Z}/2$ acts on the general linear group of a hermitian ring A as described in [10, Section 7.] and this induces an action of $\mathbb{Z}/2$ on the higher algebraic K-groups of A. There is also an action on $K_0(A)$ by hermitian duality $[P] \mapsto [P^*]$. For $x \in K_n(A)$ write $\bar{x}$ for the image of x under the action of the non-trivial element of $\mathbb{Z}/2$.

Definition 4.8. Let

$$\begin{aligned} k_n(A) =& \{x \in K_n(A) \mid x = \bar{x}\}/\{x \mid x = y + \bar{y}, \text{ for some } y \in K_n(A)\} \\ k'_n(A) =& \{x \in K_n(A) \mid x = -\bar{x}\}/\{x \mid x = y - \bar{y}, \text{ for some } y \in K_n(A)\} \end{aligned}$$

The reader who is proficient in group cohomology will notice that by definition these are Tate cohomology groups (see e.g. [56, 6.2.4])

$$\begin{aligned} k_n(A) =& \hat{H}^0(\mathbb{Z}/2; K_n(A)) \\ k'_n(A) =& \hat{H}^1(\mathbb{Z}/2; K_n(A)). \end{aligned}$$

Example 4.9. Let A be a commutative hermitian ring with $2 \in A^\times$ and trivial involution. The group $\mathbb{Z}/2$ acts on $K_1(A) \cong A^\times$ by $a \mapsto a^{-1}$, so $k_1(A)$ is the group $\mu_2(A)$ of second roots of unity of A. On the other hand $k'_1(A)$ is isomorphic to the group $A^\times/(A^\times)^2$ of square classes of $A^\times$.

Theorem 4.10: *(12-stage exact sequence) There is a natural exact sequence of groups*

$$\begin{tikzcd} k_{n+1}(A) \arrow[r] & {}_{-\varepsilon}W_{n+2}(A) \arrow[r] & {}_{\varepsilon}W'_n(A) \arrow[r] & k'_{n+1}(A) \arrow[r] & {}_{-\varepsilon}W'_{n+1}(A) \arrow[r] & {}_{-\varepsilon}W_{n+1}(A) \arrow[d] \\ {}_{\varepsilon}W_{n+1}(A) \arrow[u] & {}_{\varepsilon}W'_{n+1}(A) \arrow[l] & k_{n+1}(A) \arrow[l] & {}_{-\varepsilon}W'_n(A) \arrow[l] & {}_{\varepsilon}W_{n+2}(A) \arrow[l] & k_{n+1}(A) \arrow[l] \end{tikzcd}$$

(see [27] for the definitions of the maps and a proof of the theorem).

Example 4.11. Let $A = \mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ with trivial involution. We now determine the groups ${}_{\varepsilon}KQ_2(A)$, for $\varepsilon = 1$ and $\varepsilon = -1$.

- $\varepsilon = 1$: From the 12-stage exact sequence with $n = 0$ we get the exact sequence

  $${}_{-1}W_1(A) \longrightarrow k_1(A) \longrightarrow {}_1W_2(A) \longrightarrow {}_{-1}W'_0(A).$$

  By Example 3.17 ${}_{-1}KQ_1(A) = 0$ so the group ${}_{-1}W_1(A)$ must also be 0. The group ${}_{-1}KQ_0(A)$ is free abelian generated by the class $[H]$ of the hyperbolic plane. The forgetful map sends $[H]$ to 2 times the generator in $K_0(A) \cong \mathbb{Z}$, hence it is injective. It follows that ${}_{-1}W'_0(A) = 0$ and that the middle map in the above exact sequence is an isomorphism. From Example 4.9 we have an isomorphism $k_1(A) \cong \mathbb{Z}/2$ and by [7, A8] $K_2(A)$ is trivial. The epimorphism

  $${}_1KQ_2(A) \to {}_1W_2(A)$$

  is therefore an isomorphism, so ${}_1KQ_2(A)$ is cyclic of order 2.

- $\varepsilon = -1$: From the calculation in Example 3.7 we know that ${}_1KQ_0(A) \cong K_0(A) \oplus {}_1W_0'(A)$, where $K_0(A) \cong \mathbb{Z}$ by the rank map and ${}_1W_0'(A) \cong A^\times/(A^\times)^2$ by the discriminant map. As in [12, lemma 2.5] the 12-step exact sequence gives a short exact sequence

$$0 \longrightarrow {}_{-1}W_2(A) \longrightarrow {}_1W_0'(A) \xrightarrow{\text{disc}} A^\times/(A^\times)^2 \longrightarrow 0.$$

Since the discriminant map is an isomorphism, the group ${}_{-1}W_2(A)$ must be 0. In the exact sequence

$$K_2(A) \longrightarrow {}_{-1}KQ_2(A) \longrightarrow {}_{-1}W(A)$$

the first and the last term are trivial, hence so is ${}_{-1}KQ_2(A)$.

4.3 K-theory spectra and summary of results so far

When the ring A is a Banach algebra (with involution) over $\mathbb{C}$ or $\mathbb{R}$ there are topological versions $K^{top}(A)$ of algebraic K-theory and ${}_\varepsilon KQ^{top}(A)$ of hermitian K-theory. See [47] and [28] for an overview of these theories and how they relate to their algebraic counterparts. For $n \geq 1$ define

$$K_n^{top}(A) = \pi_{n-1}(GL(A))$$
$${}_\varepsilon KQ_n^{top}(A) = \pi_{n-1}({}_\varepsilon O(A))$$

where the groups $GL(A)$ and ${}_\varepsilon O(A)$ are topologized as the colimits of the topological groups $GL_n(A)$ and ${}_\varepsilon O_{n,n}(A)$ respectively. For $A = \mathbb{R}, \mathbb{C}$ we recover the higher K-groups of a point. These groups can be found in [30, 5.19]. For any $q \neq 0$ there are isomorphisms [52, 28]

$$K_n(\mathbb{C}, \mathbb{Z}/q) \cong K_n^{top}(\mathbb{C}, \mathbb{Z}/q)$$
$$K_n(\mathbb{R}, \mathbb{Z}/q) \cong K_n^{top}(\mathbb{R}, \mathbb{Z}/q)$$
$${}_\varepsilon KQ_n(\mathbb{C}, \mathbb{Z}/q) \cong {}_\varepsilon KQ_n^{top}(\mathbb{C}, \mathbb{Z}/q)$$
$${}_\varepsilon KQ_n(\mathbb{R}, \mathbb{Z}/q) \cong {}_\varepsilon KQ_n^{top}(\mathbb{R}, \mathbb{Z}/q)$$

obtained by comparing the corresponding groups GL and ${}_\varepsilon O$ with the discrete and usual topologies. There are corresponding isomorphisms with pro-finite coefficients. The groups $K_n^{top}(A)$ and ${}_\varepsilon KQ^{top}(A)$ satisfy Bott periodicity and are in fact the n-th homotopy groups of spectra $\mathcal{K}^{top}(A)$ and ${}_\varepsilon\mathcal{KQ}^{top}(A)$ respectively.

We now returns to the discrete case. For a discrete ring A with involution the K-theory spaces of Definition 4.3 and Definition 4.5 are infinite loop spaces and hence fit into Ω-spectra. The algebraic and ε-hermitian K-theory spectra of a hermitian ring A are defined by [12, p.3]

$$\mathcal{K}(A)_r = \begin{cases} \Omega^{r+1} BGL(SA)^+ & \text{for } r \geq 0, \\ BGL(S^{-r}A)^+ & \text{for } r < 0 \end{cases}$$

$${}_\varepsilon\mathcal{KQ}(A)_r = \begin{cases} \Omega^{r+1} B_\varepsilon O(SA)^+ & \text{for } r \geq 0, \\ B_\varepsilon O(S^{-r}A)^+ & \text{for } r < 0 \end{cases}$$

where for a natural number n the ring S^nA is the n-th suspension of A as defined in [10, App. A]. There is a (non-natural) homotopy equivalence ${}_\varepsilon\mathcal{KQ}(A)_0 \simeq {}_\varepsilon KQ_0(A) \times B_\varepsilon O(A)^+$.

The goal for this thesis at the outset was to show that there is a 2-adic equivalence (on -1-connected covers) of spectra

$$
{}_\varepsilon\mathcal{KQ}(\mathbb{Z}[\tfrac{1}{2}, \sqrt{-1}]) \simeq_2 {}_\varepsilon\mathcal{KQ}(\mathbb{F}_5) \vee \Omega({}_\varepsilon\mathcal{KQ}^{top}(\mathbb{C})). \tag{2}
$$

Having such an equivalence means having isomorphisms

$$
\mathbb{Z}_2 \otimes {}_\varepsilon KQ_i(\mathbb{Z}[\tfrac{1}{2}, \sqrt{-1}]) \cong \mathbb{Z}_2 \otimes ({}_\varepsilon KQ_i(\mathbb{F}_5) \oplus {}_\varepsilon KQ^{top}_{i+1}(\mathbb{C})),
$$

so before continuing we should make sure that such isomorphisms can exist. We have computed the groups ${}_\varepsilon KQ_i(\mathbb{Z}[\frac{1}{2}, \sqrt{-1}])$ for $i = 0, 1, 2$ and we can compare them with the groups ${}_\varepsilon KQ_i(\mathbb{F}_5)$ and ${}_\varepsilon KQ^{top}_{i+1}(\mathbb{C})$, which are already known from [19] and [30, 5.19]. The first table shows the groups for $\varepsilon = 1$.

	${}_1KQ^{top}_{i+1}(\mathbb{C})$	${}_1KQ_i(\mathbb{F}_5)$	${}_1KQ_i(\mathbb{Z}[\frac{1}{2}, \sqrt{-1}])$
$i = 0$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$
$i = 1$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$
$i = 2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$

We see that if we take the direct sum of the two columns on the left, we get the column on the right, as desired. For $i = 0, 1$ it is easy to see from the calculations of the groups ${}_1KQ_i(\mathbb{Z}[\frac{1}{2}, \sqrt{-1}])$ and ${}_1KQ_i(\mathbb{F}_5)$ that the map $\mathbb{Z}[\frac{1}{2}, \sqrt{-1}] \twoheadrightarrow \mathbb{Z}[\frac{1}{2}, \sqrt{-1}]/(1 + \sqrt{-1}) \xrightarrow{\cong} \mathbb{F}_5$ induces the maps that we would get from a decomposition as in (2).

The second table shows the shows the groups for $\varepsilon = -1$.

	${}_{-1}KQ^{top}_{i+1}(\mathbb{C})$	${}_{-1}KQ_i(\mathbb{F}_5)$	${}_{-1}KQ_i(\mathbb{Z}[\frac{1}{2}, \sqrt{-1}])$
$i = 0$	0	$\mathbb{Z}$	$\mathbb{Z}$
$i = 1$	0	0	0
$i = 2$	0	0	0

Again we see that the rightmost column is the direct sum of the other two, albeit in a trivial way. Clearly the maps induced by the map $\mathbb{Z}[\frac{1}{2}, \sqrt{-1}] \twoheadrightarrow \mathbb{Z}[\frac{1}{2}, \sqrt{-1}]/(1 + \sqrt{-1}) \xrightarrow{\cong} \mathbb{F}_5$ are the ones we want in these low degrees.

The problem now is that we don't have any candidate map between the spectra $\Omega({}_\varepsilon\mathcal{KQ}^{top}(\mathbb{C}))$ and ${}_{-1}KQ_i(\mathbb{Z}[\frac{1}{2}, \sqrt{-1}])$ to realize the desired homotopy equivalence and it is not clear how we would define any such non-trivial map. The solution will be to work with so-called étale ε-hermitian K-theory.

5 Sites, sheaves and étale cohomology

5.1 Sites and sheaves

Let X be a topological space. The open subsets of X form a category $\mathscr{T}(X)$ with inclusions as arrows. The intersection of two opens U and V in $\mathscr{T}(X)$ is their fiber product $U \times_X V$. A presheaf of (say) sets on X is a functor from $\mathscr{T}(X)^{op}$ to Set and it is a sheaf if it satisfies the following gluing condition:

- Let U be an open set of X and $\{U_i\}_{i\in\mathcal{I}}$ be an open cover of U, then the following sequence is an equalizer in Set
$$F(U) \to \prod_{i\in\mathcal{I}} F(U_i) \rightrightarrows \prod_{(i,j)\in\mathcal{I}^2} F(U_i \cap U_j).$$
In more lay terms this means that giving a section $s \in F(U)$ is equivalent to giving sections $s_i \in F(U_i)$ which agree on the intersections of the U_i's, that is, such that $s_i\big|_{U_i\cap U_j} = s_j\big|_{U_j\cap U_i}$.

This motivates the following two definitions:

Definition 5.1. [53, p.25] A site T consists of a category $cat(T)$ and a set $cov(T)$ of coverings, i.e. families of morphisms $\{U_i \to U\}_{i\in\mathcal{I}}$ satisfying the following conditions:

1. For a morphism $V \to U$ in $cat(T)$ and a covering $\{U_i \to U\}_{i\in\mathcal{I}}$ the fiber products $U_i \times_U V$ exist and $\{U_i \times_U V \to V\}_{i\in\mathcal{I}}$ is in $cov(T)$.
2. Given a covering $\{U_i \to U\}_{i\in\mathcal{I}}$ in $cov(T)$ and for each $i \in \mathcal{I}$ a covering $\{V_{ij} \to U_i\}_{j\in\mathcal{J}_i}$ in $cov(T)$ then the family of composites $\{V_{ij} \to U\}^{i\in\mathcal{I}}_{j\in\mathcal{J}_i}$ is in $cov(T)$.
3. If $f : V \to U$ is an isomorphism then $\{V \to U\}$ is in $cov(T)$.

Definition 5.2. [53, p.25] Let $\mathscr{C}$ be a category with products. A presheaf on a site T with values in $\mathscr{C}$ is a functor $F : T^{op} \to \mathscr{C}$. Presheaves satisfying the following *sheaf condition* are sheaves on T:

- For every $U \in cat(T)$ and every covering $\{U_i \to U\}_{i\in\mathcal{I}}$, the following diagram is an equalizer diagram in $\mathscr{C}$:
$$F(U) \to \prod_{i\in\mathcal{I}} F(U_i) \rightrightarrows \prod_{(i,j)\in\mathcal{I}^2} F(U_i \times_U U_j).$$

Remark 5.3. For the category $\mathscr{T}(X)$ of open sets of a topological space X these definitions specialize to the familiar ones.

The language of sites is most extensively used in algebraic geometry. A basic example is the (small) Zariski site on a scheme X, denoted by X^{Zar}. It is simply the category of open subschemes of X with the usual coverings.

5.2 The étale site on a scheme

For a scheme X and x a point in X, let $\mathfrak{m}_x$ denote the maximal ideal of the local ring $\mathcal{O}_{X,x}$ and set $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. Let $f : X \to Y$ be a morphism of schemes and let x be a point in X. If $\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$ and the map $k(f(x)) \to k(x)$ is a finite separable field extension, we say f is *unramified* at x. The map f is called unramified if it is unramified at all points of X. This clearly generalizes the ideal-theoretic notion of being unramified in number theory. A morphism $f : X \to Y$ is *flat* if for each point $x \in X$ the ring $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{Y,f(x)}$-module. It is *locally finitely presented* if for each point $x \in X$ there are open affine neighborhoods V of $f(x)$ and U of x such that $f(U) \subseteq V$ and the $\Gamma(V, \mathcal{O}_Y)$-algebra $\Gamma(U, \mathcal{O}_X)$ is finitely presented. When Y is (locally) noetherian this is the same as (locally of) finite type.

Definition 5.4. A morphism of schemes $f : X \to Y$ is called étale if it is locally finitely presented, flat and unramified. Let $\acute{\mathrm{E}}\mathrm{t}(X)$ be the subcategory of Sch/X whose objects are the étale maps $Y \to X$ and whose morphisms are the étale maps over X.

Étaleness is a very well behaved property for morphisms. It is preserved under both composition and base change. Given three morphisms f, g, h such that $f = g \circ h$, then if any two of these morphisms is étale, so is the third [53, p.84]. This implies that $\acute{\mathrm{E}}\mathrm{t}(X)$ is in fact a full subcategory of Sch/X. It follows easily from the definitions that open immersions are étale, hence any Zariski cover is also an étale cover.

Example 5.5. For smooth varieties over $\mathbb{C}$ the étale maps are precisely those that induce isomorphisms on all tangent spaces [37, p.20].

As we saw in Lemma 1.15 and the discussion following it, for any finite field extension $F/\mathbb{Q}$ there is some prime p which ramifies in $\mathcal{O}_F$. This means that the corresponding morphism of schemes $Spec(\mathcal{O}_F) \to Spec(\mathbb{Z})$ is ramified at the point corresponding to p and hence not étale. On the other hand, this is the only thing that can go wrong. The following lemma is easily verified.

Lemma 5.6: *Let F be a number field and $\mathcal{O}_F$ its ring of integers. Then the map $Spec(\mathcal{O}_F) \to Spec(\mathbb{Z})$ is étale at every point not lying over a prime dividing the discriminant Δ_F.*

The problem can be avoided by inverting the discriminant of the extension in all the involved rings. Geometrically this corresponds to working only over the open subscheme $Spec(\mathbb{Z})\setminus V(\Delta_F)$ of $Spec(\mathbb{Z})$.

Example 5.7. The discriminant of $\mathbb{Q}(\sqrt{-1})/\mathbb{Q}$ is -4 by Example 1.16, so the map $Spec(\mathbb{Z}[\frac{1}{2}, \sqrt{-1}]) \to Spec(\mathbb{Z}[\frac{1}{2}])$ is étale.

Definition 5.8. Let X be a scheme. The (small) étale site on X is denoted by $X^{\text{ét}}$ and has underlying category $\acute{\mathrm{E}}\mathrm{t}(X)$. A covering in $X^{\text{ét}}$ is a family $\{U_i \xrightarrow{f_i} U\}_{i\in\mathcal{I}}$ of étale morphisms such that $U = \bigcup_{i\in\mathcal{I}} f_i(U_i)$. Such collections of maps are called surjective étale families.

Given an X-scheme Y, the functor $\hom_{Sch/X}(-, Y) : \text{Ét}(X) \to Set$ is a sheaf on $X^{\text{ét}}$ [37, p.44]. Many of the most interesting sheaves on $X^{\text{ét}}$ arise in this way.

Example 5.9. The additive group sheaf $\mathbb{G}_a$ on $X^{\text{ét}}$ is given by sending an X-scheme $Y \to X$ to $\mathcal{O}_Y(Y)$, the additive group of global sections of the structure sheaf. There is an isomorphism of sheaves on $X^{\text{ét}}$

$$\mathbb{G}_a(-) \cong \hom_{Sch/X}(-, \mathbb{A}^1_X)$$

where $\mathbb{A}^1_X$ is defined to be $X \times_{Spec(\mathbb{Z})} \mathbb{A}^1_{\mathbb{Z}}$.

Example 5.10. The multiplicative group sheaf $\mathbb{G}_m$ sends Y to $\Gamma(Y, (\mathcal{O}_Y))^\times = \mathcal{O}_Y(Y)^\times$. Let $\mathbb{G}_{m,\mathbb{Z}} = Spec(\mathbb{Z}[x, x^{-1}])$ and define $\mathbb{G}_{m,X} = \mathbb{G}_{m,\mathbb{Z}} \times_{Spec(\mathbb{Z})} X$. There is an isomorphism of sheaves on $X^{\text{ét}}$

$$\mathbb{G}_m \cong \hom_{Sch_{/X}}(-, \mathbb{G}_{m,X})$$

Other important sheaves of this kind include the n-th roots of unity μ_n, the $n \times n$ general linear group GL_n and, importantly to us, the $2n$-th ε-orthogonal group ${}_\varepsilon O_{n,n}$ (trivial involution).

Example 5.11. Let X be a scheme. The category Sch/X of schemes over X can be given the structure of a site by letting the coverings be the surjective étale families. This is usually called the big étale site on X. The sheaves $\mathbb{G}_a$ and $\mathbb{G}_m$ are representable as functors from Sch/X to Ab. Such sheaves are called representable sheaves.

5.3 Étale cohomology

A sheaf with values in abelian groups is called an abelian sheaf. For any scheme X the category $Ab(X^{\text{ét}})$ of abelian sheaves on $X^{\text{ét}}$ is abelian and the functor $F \mapsto \Gamma(X, F) = F(X)$ is left exact [37, p.52]. A map of $F \to G$ of sheaves on $X^{\text{ét}}$ is said to be locally surjective if for every étale X-scheme U the and section $s \in G(U)$, there is an étale covering $\{U_i \to U\}$ such that for all i, the section $s\big|_{U_i}$ is in the image of $f_{U_i} : F(U_i) \to G(U_i)$.

A short exact sequence of abelian sheaves on $X^{\text{ét}}$ is a sequence

$$0 \longrightarrow F' \xrightarrow{f} F \xrightarrow{g} F'' \longrightarrow 0$$

of sheaves such that for all $U \to X$ étale, the sequence

$$0 \longrightarrow F'(U) \xrightarrow{f_U} F(U) \xrightarrow{g_U} F''(U)$$

is exact and g is locally surjective.

Definition 5.12. A geometric point of a scheme X is a map $\bar{x} : Spec(\Omega) \to X$ where Ω is a separably closed field.

Giving a geometric point on X is equivalent to giving a point x of X and an embedding of $k(x)$ in a separably closed field Ω. Let X be a scheme and $\bar{x}$ a geometric point of X. The étale covers $U \to X$ such that $\bar{x}$ factors through $U \to X$ form a cofiltering category $\bar{x} \downarrow \text{Ét}(X)$ (see Appendix A). For any sheaf F on $X^{\text{ét}}$ define the stalk of F at $\bar{x}$ by

$$F_{\bar{x}} = \operatorname*{colim}_{U \in obj(\bar{x} \downarrow \text{Ét}(X))} F(U)$$

Lemma 5.13: *([37, p.52] A sequence of sheaves*

$$0 \longrightarrow F' \xrightarrow{f} F \xrightarrow{g} F'' \longrightarrow 0$$

on $X^{\text{ét}}$ is exact if and only if the sequence

$$0 \longrightarrow F'_{\bar{x}} \xrightarrow{f_{\bar{x}}} F_{\bar{x}} \xrightarrow{g_{\bar{x}}} F''_{\bar{x}} \longrightarrow 0$$

is exact for any geometric point $\bar{x}$ of X.

Example 5.14 (Kummer sequence)**.** Let X be a scheme. The sequence of abelian sheaves on $X^{\text{ét}}$

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m$$

is exact. If, for all points $x \in X$, the number n is a unit in $(k(x))$, then the sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \longrightarrow 0,$$

called the Kummer sequence, is exact. To see this choose a geometric point $Spec(\Omega) \to X$ of X. For any element $a \in \Omega^\times = (\mathbb{G}_m)_{\bar{x}}$ the polynomial $t^n - a$ is separable, by the condition on the residue characteristics of the points of X. So, since Ω is separably closed a must have an n-th root in $\Omega^\times$, hence by varying a we see that the n-th power map $\Omega^\times \overset{(-)^n}{\to} \Omega^\times$ is surjective. Now it follows from Lemma 5.13 that the sequence of sheaves is exact.

The category $Ab(X^{\text{ét}})$ has enough injectives [53, p.53], so the left exact functor $\Gamma(X, -)$ has right derived functors.

Definition 5.15. Let X be a scheme and F an abelian sheaf on $X^{\text{ét}}$. The q-th étale cohomology group of X with coefficients in F is given by

$$H^q_{\text{ét}}(X, F) = R^q\Gamma(X, F)$$

where $R^q\Gamma(X, -)$ is the q-th right derived functor of $\Gamma(X, -)$.

Theorem 5.16 (Hilbert's Theorem 90)**:** *[53, p.107] There is a natural isomorphism*

$$H^1_{\text{ét}}(X, \mathbb{G}_m) \cong Pic(X),$$

where $Pic(X)$ denotes the Picard group of the scheme X.

There is a sheafification functor from the category of presheaves on $X^{\text{ét}}$ to the category of sheaves[37, p.54]. So, given any abelian group A we can sheafify the presheaf that is constantly A to get A_X, the *constant sheaf* of A on $X^{\text{ét}}$.

Let X be a scheme such that n is a global unit on X and $\Gamma(X, \mathcal{O}_X)$ contains a primitive n-th root of unity ζ_n. Then $1 \mapsto \zeta_n$ defines an isomorphism of sheaves $(\mathbb{Z}/n)_X \xrightarrow{\cong} \mu_n$. The Kummer sequence 5.14 can be written as

$$0 \longrightarrow (\mathbb{Z}/n)_X \longrightarrow \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \longrightarrow 0$$

which gives a long exact sequence in cohomology

$$0 \longrightarrow \mathbb{Z}/n \longrightarrow \Gamma(X, \mathcal{O}_X)^\times \xrightarrow{(-)^n} \Gamma(X, \mathcal{O}_X)^\times \longrightarrow H^1_{\text{ét}}(X, (\mathbb{Z}/n)_X) \longrightarrow$$

$$Pic(X) \xrightarrow{(-)^n} Pic(X) \longrightarrow \cdots ,$$

the $Pic(X)$-terms coming from theorem 5.16. From this we get a short exact sequence [36, p.126]

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X)^\times/(\Gamma(X, \mathcal{O}_X)^\times)^n \longrightarrow H^1_{\text{ét}}(X, (\mathbb{Z}/n)_X) \longrightarrow Pic(X)_n \longrightarrow 0,$$

where $Pic(X)_n$ denotes the n-torsion in the group $Pic(X)$.

Example 5.17. Applying this to the case $n = 2$, $A = \mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ and $X = Spec(A)$ we get an isomorphism

$$A^\times/(A^\times)^2 \cong H^1_{\text{ét}}(Spec(A), \mathbb{Z}/2)$$

since the Picard group of A is trivial. For the square classes of $A^\times$ we have the description $A^\times/(A^\times)^2 \cong \mathbb{Z}/2\{1 + \sqrt{-1}\} \oplus \mathbb{Z}/2\{\sqrt{-1}\}$ by Example 1.16.

Example 5.18. The analogous calculations for $Spec(F)$, where F is a field, show that

$$H^1_{\text{ét}}(Spec(F), \mathbb{Z}/2) \cong F^\times/(F^\times)^2.$$

In particular, when F is finite and of odd order, we have an isomorphism

$$H^1_{\text{ét}}(Spec(F), \mathbb{Z}/2) \cong \mathbb{Z}/2$$

since the group of units in a finite field is cyclic. Other examples include

$$H^1_{\text{ét}}(Spec(\mathbb{R}), \mathbb{Z}/2) \cong \mathbb{Z}/2 \quad \text{and} \quad H^1_{\text{ét}}(Spec(\mathbb{C}), \mathbb{Z}/2) = 0.$$

5.4 The étale fundamental group

Recall that a morphism of schemes $f : X \to Y$ is called finite if there is a covering of Y by affine opens $U_i \cong Spec(A_i)$ such that each $V_i = f^{-1}(U_i)$ is affine, say $V_i \cong Spec(B_i)$, and the induced maps $A_i \to B_i$ are a finite maps of rings.

Definition 5.19. Let $f\acute{\mathrm{E}}\mathrm{t}(X)$ be the full subcategory of $\acute{\mathrm{E}}\mathrm{t}(X)$ whose objects are finite étale X-schemes $Y \to X$. The finite étale site $X^{f\text{ét}}$ on X has underlying category $f\acute{\mathrm{E}}\mathrm{t}(X)$ and coverings given by surjective families.

This site is an algebraic variant of the category of (finite) covering spaces of a fixed topological space. In fact there is a general notion of Galois category due to Grothendieck of which these are special cases [33, 23].

Definition 5.20. Let $\bar{x}$ be a geometric point of a scheme X. Consider the functor $F_{\bar{x}}$ which takes a finite étale X-scheme $Y \xrightarrow{f} X$ to the finite set $\mathrm{Hom}_X(\bar{x}, Y)$. The étale fundamental group of X at $\bar{x}$ is given by

$$\pi_1^{\text{ét}}(X, \bar{x}) = Aut(F_{\bar{x}}).$$

If X is connected then any two choices of geometric point will give isomorphic fundamental groups (regardless of characteristic and codimension!) [37, p.27]. Because of this the "basepoint" $\bar{x}$ is often suppressed in the notation.

Example 5.21. When $X = Spec(F)$ for F a field and $\bar{x} : Spec(F_{sep}) \to Spec(F)$ is a closed point corresponding to a separable closure F_{sep} of F there is an isomorphism [36, I,5.2.(a)]

$$\pi_1^{\text{ét}}(X, \bar{x}) \cong \mathrm{Gal}(F_{sep}/F).$$

Examples include

$$\pi_1^{\text{ét}}(Spec(\mathbb{R})) \cong \mathbb{Z}/2 \quad \text{and} \quad \pi_1^{\text{ét}}(Spec(\mathbb{C})) = 0.$$

Example 5.22. ([33, p.96])Let $X = Spec(\mathcal{O}_F[\frac{1}{a}])$ for some number field F and algebraic integer $a \in \mathcal{O}_F$. Write M for the maximal algebraic extension of F which is only ramified at the primes dividing a. There is an isomorphism $\pi_1^{\text{ét}}(X) \cong \mathrm{Gal}(M/F)$. An object of $f\acute{\mathrm{E}}\mathrm{t}(X)$ is (up to isomorphism) of the form

$$\coprod_{i=1}^{n} Spec(B_i) \to X$$

where B_i is the integral closure of $\mathcal{O}_F[\frac{1}{a}]$ in some finite subextension K_i of M.

6 Étale homotopy theory

In this section we will introduce simplicial schemes and try to motivate étale homotopy theory. These tools will be used in the next section to prove the desired homotopy equivalence (2).

6.1 Simplicial schemes

A simplicial scheme is a functor $X : \Delta^{op} \to Sch$, where Δ denotes the finite ordinal category (see [21] for a definition of this category). The category $SSch$ of simplicial schemes is the functor category $Sch^{\Delta^{op}}$, whose morphisms are natural transformations of functors. The value of the simplicial scheme X on the object $[n]$ of Δ is denoted by X_n. A simplicial scheme can be thought of as a sequence of schemes X_n together with degeneracy morphisms $s_i : X_n \to X_{n+1}$ and face morphisms $d_i : X_n \to X_{n-1}$ for $0 \leq i \leq n$, satisfying the usual simplicial identities [21, p.4].

Given a simplicial set T and a simplicial scheme X one can form the simplicial scheme $X \otimes T$ which is given in degree n by the disjoint union $(X \otimes T)_n = \coprod_{t \in T_n} X_t$. The i-th degeneracy morphism $s_i : (X \otimes T)_n \to (X \otimes T)_{n+1}$ is given on the summand corresponding to $t \in T_n$ by applying the i-th degeneracy map s_i^X of X_n and mapping to the summand corresponding to $s_i^T(t) \in T_{n+1}$. In formulas:

$$s_i\big|_{X_{n,t}} = s_i^X : X_{n,t} \to X_{n+1,s_i(t)}.$$

The face morphisms are defined analogously. The functor sending a scheme X to the constant simplicial scheme, all of whose structure morphisms are identity morphisms, is a full embedding of Sch in $SSch$. We will write simply X for this constant simplicial scheme.

Example 6.1. Let X be a scheme and G a group scheme over X (see e.g. [54, Ch.3] for a definition). For all $n \geq 0$ define $BG_n = G^{\times_X^n}$. Let $s_i : BG_n \to BG_{n+1}$ be the composite

$$G^{\times_X^n} \xrightarrow{\cong} G^{\times_X^i} \times_X X \times_X G^{\times_X^{n-i}} \xrightarrow{id_G^i \times e \times id_G^{n-i}} G^{\times_X^{n+1}} ,$$

where $e : X \to G$ is the unit map of G. Similarly let $d_i : BG_n \to BG_{n-1}$, for $1 < i < n$ be the morphism

$$G^{\times_X^n} \xrightarrow{id_G^{i-1} \times \mu \times id_G^{n-i}} G^{\times_X^{n-1}}$$

where $\mu : G \times_X G \to G$ is the multiplication on G. When $i = 1$ or $i = n$ the map d_i forgets the i-th factor in the fiber product. The schemes BG_n and morphisms s_i, d_j form a simplicial scheme called the classifying scheme of G over X.

The next lemma is easily verified.

Lemma 6.2: *Let X be a scheme and Y a simplicial scheme. For each n, the natural map*

$$SSch(X \otimes \Delta^n, Y) \overset{\cong}{\to} Sch(X, Y_n)$$

given by

$$f \mapsto f\big|_{id_{\Delta^n}} : X \to Y_n$$

is a bijection.

Definition 6.3. Let Z be a simplicial scheme and let $SSch/Z$ denote the category of simplicial schemes over Z. For X and Y are in $SSch/Z$ the mapping complex $\mathrm{Hom}(X,Y)_Z$ is the simplicial set given in degree n by

$$(\mathrm{Hom}(X,Y)_Z)_n = SSch/Z(X \otimes \Delta^n, Y).$$

Maps $\alpha : [m] \to [n]$ in Δ induce maps of simplicial sets $\Delta^m \to \Delta^n$ and therefore also a maps

$$SSch/Z(X \otimes \Delta^n, Y) \to SSch/Z(X \otimes \Delta^m, Y)$$

by precomposition. These induced maps are the structure maps of $\mathrm{Hom}(X,Y)_Z$.

Let R be a ring, $X = Spec(R)$, G a group scheme over $Spec(R)$ and A an R-algebra. By Lemma 6.2 we get a natural isomorphism

$$(\mathrm{Hom}(Spec(A), BG)_X)_n = SSch_{/X}(Spec(A) \otimes \Delta^n, BG) \xrightarrow{\cong} Sch_{/X}(Spec(A), BG_n).$$

The set on the right is by definition $Sch_{/X}(Spec(A), G^{\times^n_R})$, which is naturally isomorphic to the n-fold cartesian product of sets $G(A) \times \cdots \times G(A)$. Hence by naturality $\mathrm{Hom}(Spec(A), BG)_X$ is the ordinary bar construction [56, 8.1.7] on the group $G(A)$.

Example 6.4. The main examples of interest to us are when $G = GL_n$ or when $G = {}_\varepsilon O_{n,n}$. The isomorphisms

$$BGL_n(A) \cong \mathrm{Hom}(Spec(A), BGL_n)_{Spec(\mathbb{Z})}$$

and

$$B_\varepsilon O_{n,n}(A) \cong \mathrm{Hom}(Spec(A), B_\varepsilon O_{n,n})_{Spec(\mathbb{Z})}$$

will allow us to formulate questions about K-theory and hermitian K-theory in terms of algebraic geometry.

6.2 Hypercovers and étale homotopy types

Simplicial schemes arise naturally when we consider coverings of schemes and the combinatorics of iterated intersections or fiber products of the schemes in the covers. Let X be a scheme and $\mathcal{U} = \{U_i \to X\}_{i \in \mathcal{I}}$ an étale covering. Forming the scheme $U = \coprod_i U_i$ we can realize $\mathcal{U}$ as a single étale X-scheme $U \to X$. The Čech nerve of $\mathcal{U}$ is the simplicial scheme $N(\mathcal{U})$ given in degree n by

$$N_n(\mathcal{U}) = U \times_X U \times_X \cdots \times_X U,$$

the $n+1$-fold fiber product of U over X. The boundary morphisms are projections, and degeneracies are products of identity maps with suitable diagonals. If the maps $U_i \to X$ are Zariski open immersions the X-scheme $U^{(\times_X)^{n+1}} \to X$ is isomorphic to a disjoint union of the immersions $U_{i_1} \cap \cdots \cap U_{i_n} \hookrightarrow X$ varying over all $(i_1, \ldots, i_n) \in \mathcal{I}^n$. The restriction of the map $d_j : N_n(\mathcal{U}) \to N_{n-1}(\mathcal{U})$ to $U_{i_1} \cap \cdots \cap U_{i_n}$ equals the inclusion

$$U_{i_1} \cap \cdots \cap U_{i_n} \hookrightarrow U_{i_1} \cap \cdots \cap \hat{U}_{i_j} \cap \cdots \cap U_{i_n},$$

where the $\hat{U}_{ij}$ means that the ij-term is omitted from the intersection. In this way the Čech nerve of a (not necessarily Zariski) covering is an object that contains all iterated "intersections" of elements in the cover and knows how they are glued together.

Let F be an abelian sheaf on $X^{\text{ét}}$. Since $N(\mathcal{U})$ is a simplicial scheme we can evaluate F on it and get a cosimplicial abelian group. Taking the alternating sum of the coface maps gives a cochain complex

$$\check{\mathcal{C}}^{\bullet}(\mathcal{U}, F) = \big(F(U) \longrightarrow F(U \times_X U) \longrightarrow F(U \times_X U \times_X U) \longrightarrow \cdots\big)$$

where by the sheaf condition for the étale cover $U \to X$ the zeroth cohomology group $H^0(\check{\mathcal{C}}^{\bullet}(\mathcal{U}, F))$ is naturally isomorphic to $F(X)$. We write $\check{H}^p(\mathcal{U}, F)$ for the p-th cohomology group of $\check{\mathcal{C}}^{\bullet}(\mathcal{U}, F)$ and call it the p-th Čech cohomology group of $\mathcal{U}$ with coefficients in F. Since U is actually a disjoint union of U_is over X, the complex $\check{\mathcal{C}}^{\bullet}(\mathcal{U}, F)$ can also be written as

$$\textstyle\prod_{i\in\mathcal{I}} F(U_i) \longrightarrow \prod_{(i,j)\in\mathcal{I}^2} F(U_i \times_X U_j) \longrightarrow \prod_{(i,j,k)\in\mathcal{I}^3} F(U_i \times_X U_j \times_X U_k) \longrightarrow \cdots$$

which is almost the same as the Čech-complex of the cover $\mathcal{U}$ with coefficients in F as defined in [24, p.218][2]. Čech cohomology can under nice conditions be used to compute the ordinary sheaf cohomology [36, p.99] but one must then pass to the limit over all covers. For complete generality one must work with the more general notion of hypercovers. An étale hypercover U of X is a simplicial scheme with $U_0 \to X$ an étale cover. Further, we require that for each n the scheme U_n is étale over the fiber products of the schemes U_k for $0 \le k \le n$ (see [1, Ch.8] for details). The Čech nerve of a cover of X is a hypercover; indeed one can think of a hypercover U as a refinement of the Čech nerve $N(U_0)$. For a hypercover U of X and an abelian sheaf F on $X_{\text{ét}}$ there is a cohomological descent spectral sequence

$$E_1^{pq} = H^q_{\text{ét}}(U_p, F) \Rightarrow H^{p+q}_{\text{ét}}(X, F)$$

which is natural in U and F. Furthermore, there is a natural isomorphism

$$H^q_{\text{ét}}(X, F) \cong \operatorname*{colim}_{U\in obj(HR(X))} H^q(F(U)) \tag{3}$$

where $HR(X)$ denotes the homotopy category of étale hypercovers of X [1, p.105] and $F(U)$ is the cosimplicial abelian group we get by evaluating a sheaf F on a hypercover U. The idea of étale homotopy is to work, not with X itself, but instead the homotopy category $HR(X)$ of hypercovers of X, since by the isomorphism (3) this category contains all the cohomological information of X. Taking the connected components of the objects of $HR(X)$ gives a pro-object of simplicial sets which remembers the cohomological information of the locally constant sheaves on X. This is done in the book [1] and leads to interesting computations. See the appendix A for a discussion of pro-objects and their properties. The ideas of [1] are taken even further in [20] where they are

[2]The difference is that here the products are taken over all tuples of indeces, not just increasing sequences of them.

applied to simplicial schemes. This is the machinery that has been most used for K-theory and we will use it to study ε-hermitian K-theory.

Friedlander [20, Ch.4] introduced the functor

$$(-)_{\text{ét}} : \{\text{ locally noetherian simplicial schemes}\} \to \text{pro}-SSet$$

sending a simplicial scheme X to its *étale topological type* $X_{\text{ét}}$. The construction of the functor is roughly as follows (for a full treatment see [loc. cit.]). For a simplicial scheme X one has a notion of hypercovers, which are bisimplicial schemes with good covering properties. One considers the homotopy category $HRR(X)$ of so called rigid étale hypercovers of X. This category is cofiltering and by applying the composite of the diagonal functor and the Zariski connected component functor one gets the pro-simplicial set $X_{\text{ét}} : HRR(X) \to SSet$. Given a map $X \to Y$ of simplicial schemes one can pull back hypercovers $U \to Y$ to hypercovers $X \times_Y U \to X$ and the projection maps $X \times_Y U \to U$ combine to give a strict map of pro-simplicial sets.

Several of the algebro-geometric invariants of the simplicial scheme X are reflected in the homotopy invariants of the pro-simplicial set $X_{\text{ét}}$. There is a natural one to one correspondence between the locally constant étale sheaves on X and local coefficient systems on $X_{\text{ét}}$ (see [20, p.49] for a definition of local system on $X_{\text{ét}}$ and a proof of the statement). By abuse of notation we write $A_{\text{ét}}$ for $Spec(A)_{\text{ét}}$ when A is a ring. For the definitions of homotopy invariants for pro-simplicial sets the reader is referred to the definitions A.11 and A.12 in the appendix.

Proposition 6.5: *[18, p.3]Let $\bar{a} : spec(k) \to spec(A)$ be a geometric point of a normal R-algebra A. There are natural isomorphisms:*

1. $\pi_1(A_{\text{ét}}, \bar{a}) \cong \pi_1^{\text{ét}}(Spec(A), \bar{a})$
2. $H^*(A_{\text{ét}}, \tilde{C}) \cong H^*_{\text{ét}}(spec(A), C)$ *where C is a locally constant abelian sheaf and $\tilde{C}$ is the corresponding local system on $A_{\text{ét}}$.*

Example 6.6. Let k be a field. Then by Example 5.21 and Proposition 6.5 there is an isomorphism $\pi_1(k_{\text{ét}}) \cong \mathrm{Gal}(k_{sep}/k)$, where k_{sep} is a separable closure of k. In fact, there is a homotopy equivalence of pro-simplicial sets $k_{\text{ét}} \simeq K(\mathrm{Gal}(k_{sep}/k), 1)$ [42]. This means that $\mathbb{C}_{\text{ét}}$ is contractible, $\mathbb{R}_{\text{ét}}$ is weakly equivalent to $\mathbb{R}P^\infty$ and $\mathbb{F}_q$ is a $K(\hat{\mathbb{Z}}, 1)$.

6.3 Étale K-theories

Let V be a pro-simplicial set and let S and T be pro-simplicial sets over V where $T \to V$ is a strict map. Fix a prime number ℓ. Dwyer and Friedlander [16] defined an ℓ-adic relative mapping space $Hom_\ell(S, T)_V$ of functions over V. It is functorial in the left variable and functorial for strict maps in the right variable. When $V = Z_{\text{ét}}$, $S = X_{\text{ét}}$ and $T = Y_{\text{ét}}$ for simplicial schemes X and Y over the simplicial scheme Z there is a natural map

$$\mathrm{Hom}(X, Y)_Z \to \mathrm{Hom}_\ell(X_{\text{ét}}, Y_{\text{ét}})_{Z_{\text{ét}}}. \tag{4}$$

Taking $Y = BG$ for G a group scheme over Z we define

$$BG^{\ell}(X_{\text{ét}})_Z = \mathrm{Hom}^0_{\ell}(X_{\text{ét}}, BG_{\text{ét}})_{Z_{\text{ét}}} \tag{5}$$

where the superscript 0 means that we take the connected component of the image of the map $X \to Z \xrightarrow{e} G$. This map also serves as the base point. Now let $X = Spec(\mathcal{O}_F[\frac{1}{\ell}])$, $Z = Spec(R)$ with $R = \mathbb{Z}[\frac{1}{\ell}]$ and let $Y = BGL_n$. By Example 6.4, $\mathrm{Hom}(Spec(\mathcal{O}_F[\frac{1}{\ell}]), BGL_n)_{Spec(R)}$ is naturally isomorphic to $BGL_n(\mathcal{O}_F[\frac{1}{\ell}])$, so there is a natural map

$$\chi_n : BGL_n(\mathcal{O}_F[\tfrac{1}{\ell}]) \to BGL_n^{\ell}(\mathcal{O}_F[\tfrac{1}{\ell}]_{\text{ét}})_{R_{\text{ét}}}.$$

For all $n \geq 1$ the closed immersions $GL_n \to GL_{n+1}$ induce maps

$$BGL_n^{\ell}(\mathcal{O}_F[\tfrac{1}{\ell}]_{\text{ét}})_{R_{\text{ét}}} \to BGL_{n+1}^{\ell}(\mathcal{O}_F[\tfrac{1}{\ell}]_{\text{ét}})_{R_{\text{ét}}}$$

which are compatible with the maps χ_n. Taking the colimit of this sequence of maps we get a map

$$\chi : BGL(\mathcal{O}_F[\tfrac{1}{\ell}]) \to BGL^{\ell}(\mathcal{O}_F[\tfrac{1}{\ell}]_{\text{ét}})_{R_{\text{ét}}}.$$

between the colimits of the spaces. When ℓ is odd and regular or when $\ell = 2$ and $\sqrt{-1} \in F$, the ℓ-adic Quillen-Lichtenbaum-conjecture [17] states the map χ induces an isomorphism on mod ℓ cohomology. This map factors through the plus-construction $BGL(\mathcal{O}_F[\frac{1}{\ell}])^+$ [16, 4.4] and another form of the Quillen-Lichtenbaum-conjecture states that the map

$$\chi^+ : BGL(\mathcal{O}_F[\tfrac{1}{\ell}])^+ \to BGL^{\ell}(\mathcal{O}_F[\tfrac{1}{\ell}]_{\text{ét}})_{R_{\text{ét}}}$$

induces isomorphisms on ℓ completed homotopy groups π_n for n sufficiently large.

Definition 6.7. Let A be a noetherian R-algebra of finite mod ℓ étale cohomological dimension. The ℓ-adic étale K-theory space of A is defined by

$$K^{\text{ét}}(A)_{\ell} = BGL^{\ell}(A_{\text{ét}})_{R_{\text{ét}}}.$$

Further, for $i \geq 1$ the i-th étale K-group of R is defined by

$$K_i^{\text{ét}}(A)_{\ell} = \pi_i(BGL^{\ell}(A_{\text{ét}})_{R_{\text{ét}}})$$

Much work has been done on the Quillen-Lichtenbaum conjectures. Rognes and Weibel proved [45] that when $\sqrt{-1} \in F$ the maps

$$\chi_*^+ : K_n(\mathcal{O}_F[\tfrac{1}{2}]) \otimes \mathbb{Z}_2 \to K^{\text{ét}}(\mathcal{O}_F[\tfrac{1}{2}])_2$$

are isomorphisms for $n \geq 1$. The result was extended to number fields with a real embedding by Østvær in [43]. For such number fields the definition of étale K-theory has to be modified because the ring $\mathcal{O}_F[\frac{1}{2}]$ has infinite mod 2 étale cohomological dimension [17].

From now on we will work only with $\ell = 2$, so R will denote $\mathbb{Z}[\frac{1}{2}]$. The constructions that were done for the general linear group above can also be carried out with the ε-orthogonal group schemes when $\varepsilon = 1$ or -1 and the involution is trivial. Let F be a number field with $\sqrt{-1} \in F$. The conjecture that the map

$$_\varepsilon\chi : B_\varepsilon O(\mathcal{O}_F[\tfrac{1}{2}])^+ \to B_\varepsilon O^2(\mathcal{O}_F[\tfrac{1}{2}]_{\text{ét}})_{R_{\text{ét}}}.$$

induces isomorphisms on 2-completed homotopy groups will be called the 2-adic hermitian Quillen-Lichtenbaum conjecture for $\mathcal{O}_F$.

Definition 6.8. Let A be a noetherian R-algebra. The 2-adic étale ε-hermitian K-theory space of A is defined by

$$KQ^{\text{ét}}(A)_2 = B_\varepsilon O^2(A_{\text{ét}})_{R_{\text{ét}}}.$$

Further, for $i \geq 1$ the i-th étale ε-hermitian K-group of A is defined by

$$KQ_i^{\text{ét}}(A)_2 = \pi_i(B_\varepsilon O^2(A_{\text{ét}})_{R_{\text{ét}}}).$$

7 Computations

From now on the constant ε will always equal 1 or -1. In this section we will compute the homotopy type of the ε-hermitian K-theory space of the Gaussian 2-integers and some related rings. The arguments given here work for general "2-regular" rings as defined in [42], but we will not work in this generality. We begin by fixing some notation. Let $\nu \geq 2$, ζ_{2^ν} be primitive 2^ν-th root of unity, $F = \mathbb{Q}(\zeta_{2^\nu})$, $R = \mathbb{Z}[\frac{1}{2}]$ and let A be $R[\zeta_{2^\nu}]$, the integral closure of R in F. Write c for the number of pairs of complex embeddings of F. We choose a residue field $\mathbb{F}$ of A of order p, where p is a prime which is congruent to 5 modulo 8 and totally split in F. This is always possible by the Chebotarev density theorem [41, 13.4, VII]. By "space" we mean simplicial set unless otherwise is explicitly stated. The circle S^1 means the simplicial circle $\Delta^1/\partial\Delta^1$.

This section is organized as follows: First we give an outline of a construction given in [42] of a pro-simplicial set M and a map $\alpha : M \to A_{\text{ét}}$ such that the homotopy type of the 2-adic étale K-theory space of A can be computed in terms of M. Then we prove some technical lemmas about M and α. Using these lemmas we compute the homotopy type of the étale ε-hermitian K-theory space of A. After this we go on to étale hermitian K-theory spectra and show that for A as defined above the étale ε-hermitian K-theory spectrum is 2-adically equivalent to the ordinary ε-hermitian K-theory spectrum.

By Dirichlet's unit theorem (Theorem 1.13) there is an isomorphism $A^\times \cong \mu_{2^\nu} \times \mathbb{Z}^c$, and since there is no 2-torsion in the Picard group of A [55, p.187], the Kummer sequence (Example 5.14) gives an isomorphism $H^1(A_{\text{ét}}; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{c+1}$. From a computation in [42] the cohomology groups vanish in degree 2 and higher, so we have the description

$$H^i_{\text{ét}}(A, \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & \text{if } i = 0, \\ (\mathbb{Z}/2)^{c+1} & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By Example 5.22 the group $\pi_1(A_{\text{ét}})$ is isomorphic to the Galois group of the maximal extension of F which is only ramified at 2. This group is isomorphic to a free product $\mathbb{Z}_2^{*^{c+1}}$ [22, Th.2]. As in [42] we can take one of the generators to be the Frobenius element from $\mathbb{F}$. Let $W = \vee_{i=1}^c S^1$ and let $f : W \to A_{\text{ét}}$ be a wedge of representing maps for the other generators of $\pi_1(A_{\text{ét}})$. Then the map

$$\alpha : M = \mathbb{F}_{\text{ét}} \vee W \to A_{\text{ét}} \tag{6}$$

induces an isomorphism on mod 2 cohomology. As in [42] the map f can be chosen such that the composite $W \xrightarrow{f} A_{\text{ét}} \to R_{\text{ét}}$ is homotopically trivial. We think of M as a model for $A_{\text{ét}}$. The facts that α is a mod 2 cohomology isomorphism and behaves well on π_1 will allow us to compute the étale ε-hermitian K-theory in terms of the components W and $\mathbb{F}_{\text{ét}}$ of M.

Let $\Lambda = Aut(\mu_{2^\infty})$ and let $\mathbb{Z}/2^n(k)$, for $k \geq 0$, be the local system on $K(\Lambda, 1)$ given by the Λ-module $\mu_{2^n}^{\otimes k}$, with the diagonal action of Λ on the tensor product. There is a map $\pi_1(R_{\text{ét}}) \to \Lambda$ arising from the action of $\pi_1(R_{\text{ét}})$ on the 2-adic roots of unity. This gives a map from $R_{\text{ét}}$ to $K(\Lambda, 1)$. The local system $\mathbb{Z}/2^n(k)$ pulls back via this map to $R_{\text{ét}}$, $A_{\text{ét}}$ and M. From Proposition 6.5 it follows that on the spaces arising from schemes, the local system $\mathbb{Z}/2^n(k)$ gives the same cohomology as the local system induced by the étale sheaf with the same name (see e.g. [8] for the properties of this sheaf). Since the groups $\pi_1(A_{\text{ét}})$ and $\pi_1(M)$ act on $\mathbb{Z}/2^n(k)$ through the maps to Λ, the exact sequence of local systems

$$0 \longrightarrow \mathbb{Z}/2^n(k) \longrightarrow \mathbb{Z}/2^{n+1}(k) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

on $A_{\text{ét}}$ gives rise to maps of long exact sequences

$$\begin{array}{ccccccc} \cdots \longrightarrow H^{i-1}(A_{\text{ét}}; \mathbb{Z}/2) & \longrightarrow & H^i(A_{\text{ét}}; \mathbb{Z}/2^n(k)) & \longrightarrow & H^i(A_{\text{ét}}; \mathbb{Z}/2^{n+1}(k)) \longrightarrow \cdots \\ \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow H^{i-1}(M; \mathbb{Z}/2) & \longrightarrow & H^i(M; \mathbb{Z}/2^n(k)) & \longrightarrow & H^i(M; \mathbb{Z}/2^{n+1}(k)) \longrightarrow \cdots . \end{array} \tag{7}$$

The maps for $\mathbb{Z}/2$ coefficients are isomorphisms, so by induction on n and the 5-lemma they are isomorphisms for all n.

On the étale site of $Spec(R)$ there is a pro-sheaf

$$\cdots \longrightarrow \mathbb{Z}/2^{n+1}(k) \longrightarrow \mathbb{Z}/2^n(k) \longrightarrow \cdots \longrightarrow \mathbb{Z}/2. \tag{8}$$

Define the (twisted) 2-adic cohomology of a scheme X over R by

$$H^i(X, \mathbb{Z}_2(k)) = \varprojlim_n H^i_{\text{ét}}(X, \mathbb{Z}/2^n(k)).$$

The pro-sheaf (8) induces a pro-object of local systems $\mathbb{Z}/2^n(k)$ on $K(\Lambda,1)$ and for a pro-space S over $K(\Lambda,1)$ we define the continuous (twisted) 2-adic cohomology of S by

$$H^i_c(S,\mathbb{Z}_2(k)) = \varprojlim_n H^i(S,\mathbb{Z}/2^n(k)). \tag{9}$$

The above discussion is summarized in the following lemma.

Lemma 7.1: *For any $i,k,n \geq 0$ the map $\alpha : M \to A_{\text{ét}}$ of* (6) *induces isomorphisms on cohomology with $\mathbb{Z}/2^n(k)$ coefficients and on continuous cohomology with $\mathbb{Z}_2(k)$ coefficients.*

Let $\bar{\mathbb{F}}$ be an algebraic closure of the field $\mathbb{F}$. Taking the fiber of the geometric point determined by $\bar{\mathbb{F}}$ yields a sequence

$$B_\varepsilon O_{n,n,\bar{\mathbb{F}}} \longrightarrow B_\varepsilon O_{n,n,R} \longrightarrow Spec(R)$$

of simplicial schemes. Applying the étale topological type functor and then the fiberwise Bousfield-Kan 2-completion [16, p.250] gives a fiber sequence

$$\{(\mathbb{Z}/2)_s(B_\varepsilon O_{n,n,\bar{\mathbb{F}},\text{ét}})\} \longrightarrow \{(\mathbb{Z}/2)^\bullet_s B_\varepsilon O_{n,n,R,\text{ét}}\} \longrightarrow R_{\text{ét}},$$

as in [15]. By [20, ch. 8] there is a weak equivalence of pro-spaces

$$BO^2_{n,n}(\mathbb{C}) \simeq \{(\mathbb{Z}/2)_s(B_\varepsilon O_{n,n,\bar{\mathbb{F}},\text{ét}})\}_s$$

where $BO^2_{n,n}(\mathbb{C})$ denotes the Bousfield Kan 2-completion of the simplicial set $Sing(BO_{n,n}(\mathbb{C}))$, the singular simplices of classifying space of the complex Lie group $O_{n,n}(\mathbb{C})$. For ease of notation we write F_n for the pro-space $\{(\mathbb{Z}/2)_s(B_\varepsilon O_{n,n,\bar{\mathbb{F}},\text{ét}})\}_s$. In the next lemma the spaces are topological spaces, not simplicial sets.

Lemma 7.2: *For each $n \geq 1$ the map $B_\varepsilon O_{n,n}(\mathbb{C}) \to B_\varepsilon O_{n+1,n+1}(\mathbb{C})$ arising from the inclusion of groups ${}_\varepsilon O_{n,n}(\mathbb{C}) \hookrightarrow {}_\varepsilon O_{n+1,n+1}(\mathbb{C})$ induces isomorphisms on homotopy groups π_j for $0 \leq j \leq 2n$ when $\varepsilon = 1$ and for $0 \leq j \leq 4n+1$ when $\varepsilon = -1$.*

Proof. The inclusion map ${}_\varepsilon O_{n,n}(\mathbb{C}) \hookrightarrow {}_\varepsilon O_{n+1,n+1}(\mathbb{C})$ induces isomorphisms on homotopy groups π_j for $0 \leq j \leq 2n-1$ when $\varepsilon = 1$ and for $0 \leq j \leq 4n+2$ when $\varepsilon = -1$ [25, p.82]. For each n there is a natural homotopy equivalence $\Omega B_\varepsilon O_{n,n}(\mathbb{C}) \simeq O_{n,n}(\mathbb{C})$, so we get isomorphisms on homotopy groups of the classifying spaces in degrees one less than the isomorphisms on the groups. □

Now we can prove a key technical lemma.

Lemma 7.3: *For each $n \geq 1$ there is a commuting square*

$$\begin{array}{ccc} B_\varepsilon O^2_{n,n}(M)_{R_{\text{ét}}} & \longrightarrow & B_\varepsilon O^2_{n,n}(A_{\text{ét}})_{R_{\text{ét}}} \\ \downarrow & & \downarrow \\ B_\varepsilon O^2_{n+1,n+1}(M)_{R_{\text{ét}}} & \longrightarrow & B_\varepsilon O^2_{n+1,n+1}(A_{\text{ét}})_{R_{\text{ét}}} \end{array} \tag{10}$$

where the horizontal maps are induced by the map $\alpha : M \to A_{\text{ét}}$ of (6) *and the vertical maps are induced by the closed immersion ${}_\varepsilon O_{n,n} \to {}_\varepsilon O_{n+1,n+1}$. The maps in the square induce isomorphisms on homotopy groups π_j for $0 \leq j \leq 2n$ when $\varepsilon = 1$ and for $0 \leq j \leq 4n+1$ when $\varepsilon = -1$.*

Proof. The commutativity of the square follows from the functoriality of the mapping space constructions and the fact that the immersion ${}_\varepsilon O_{n,n} \to {}_\varepsilon O_{n+1,n+1}$ of group schemes induces a strict map on étale topological types.

For a pro-space Y over $R_{\text{ét}}$ there is a natural spectral sequence [16, 2.11]

$$E_2^{p,q} = H^p_c(Y, \pi_{-q}F_n) \implies \pi_{-(p+q)}(B_\varepsilon O^2_{n,n}(Y)_{R_{\text{ét}}}). \tag{11}$$

The group Λ acts on the homotopy groups of F_n and the cohomology is continuous local cohomology (see [16, 2.8]), which generalizes (9), and it is twisted by this action of Λ. Since there are natural isomorphisms of pro-groups $\pi_{-q}F_k \simeq \pi_{-q}(BO^2_{k,k,\mathbb{C}})$ for all k and the latter groups stabilize by Lemma 7.2, the coefficient groups $\pi_{-q}F_n$ also stabilize in the same ranges. These stable groups can be found in the table [11, 6.11,(1)]. Depending on the value of q, $H^p(Y, \pi_{-q}F)$ is either 0 or isomorphic to $H^p(Y, \mathbb{Z}/2)$ or $H^p_c(Y, \mathbb{Z}_2(k))$, for some k.

Now we return to the square (10). The spaces M and $A_{\text{ét}}$ have mod 2 cohomological dimension equal to 1. Hence the E_2^{pq}-terms of the spectral sequences (11) for these spaces are zero except in the two columns where $p = 0$ and $p = 1$. It follows that all the differentials on the E_2-pages and onward are zero. By Lemma 7.1 the horizontal maps in the square, which are induced by α, yield isomorphisms on cohomology with the coefficients from the table in [11]. The maps of spectral sequences induced by the maps in the square are therefore isomorphisms on E_2^{pq}-terms with $-2n \leq q \leq 0$ when $\varepsilon = 1$ and for $-(4n+1) \leq q \leq 0$ when $\varepsilon = -1$. From these isomorphisms and the vanishing of the differentials we conclude that the maps on homotopy groups π_j are isomorphisms for $0 \leq j \leq 2n$ when $\varepsilon = 1$ and for $0 \leq j \leq 4n+1$ when $\varepsilon = -1$. $\square$

Now letting $n \to \infty$ we get:

Corollary 7.4: *The map $\alpha : M \to A_{\text{ét}}$ induces a weak equivalence*

$$B_\varepsilon O^2(M)_{R_{\text{ét}}} \to B_\varepsilon O^2(A_{\text{ét}})_{R_{\text{ét}}}$$

Let $*$ be a one point pro-space and choose a map from $*$ to the path connected space $\mathbb{F}_{\text{ét}}$. There is a commuting square

$$\begin{tikzcd} * \arrow[r] \arrow[d] & W \arrow[d] \\ \mathbb{F}_{\text{ét}} \arrow[r] & A_{\text{ét}} \end{tikzcd}$$

over $R_{\text{ét}}$, where the upper map is the inclusion of the base point and the map from the pushout of the upper left part of the diagram to $A_{\text{ét}}$ is the map $\alpha : M \to A_{\text{ét}}$. Applying the functor

$B_\varepsilon O^2(-)_{R_{\text{ét}}}$ we get a square

$$\begin{array}{ccc} B_\varepsilon O^2_{n,n}(A_{\text{ét}})_{R_{\text{ét}}} & \longrightarrow & B_\varepsilon O^2_{n,n}(W)_{R_{\text{ét}}} \\ \downarrow & & \downarrow \\ B_\varepsilon O^2_{n,n}(\mathbb{F}_{\text{ét}})_{R_{\text{ét}}} & \longrightarrow & B_\varepsilon O^2_{n,n}(*)_{R_{\text{ét}}}. \end{array} \tag{12}$$

Theorem 7.5: *1. The square*

$$\begin{array}{ccc} B_\varepsilon O^2(A_{\text{ét}})_{R_{\text{ét}}} & \longrightarrow & B_\varepsilon O^2(W)_{R_{\text{ét}}} \\ \downarrow & & \downarrow \\ B_\varepsilon O^2(\mathbb{F}_{\text{ét}})_{R_{\text{ét}}} & \longrightarrow & B_\varepsilon O^2(*)_{R_{\text{ét}}} \end{array} \tag{13}$$

obtained by taking the colimit over n in (12) *is homotopy cartesian.*

2. *Let $B_\varepsilon O^2(\mathbb{C})$ denote the Bousfield-Kan 2-completion of the singular simplices $Sing(B_\varepsilon O^2(\mathbb{C}))$ of the classifying space of the topological group ${}_\varepsilon O^2(\mathbb{C})$. There are weak equivalences $(B_\varepsilon O^2(\mathbb{C}))^W \simeq B_\varepsilon O^2(W)_{R_{\text{ét}}}$ and $B_\varepsilon O^2(\mathbb{C}) \simeq B_\varepsilon O^2(*)_{R_{\text{ét}}}$ and the square* (13) *is weakly equivalent to*

$$\begin{array}{ccc} B_\varepsilon O^2(A_{\text{ét}})_{R_{\text{ét}}} & \longrightarrow & (B_\varepsilon O^2(\mathbb{C}))^W \\ \downarrow & & \downarrow \\ B_\varepsilon O^2(\mathbb{F}_{\text{ét}})_{R_{\text{ét}}} & \longrightarrow & B_\varepsilon O^2(\mathbb{C}) \end{array}$$

where the right hand vertical map is evaluation at the basepoint.

Proof. 1) The map from $B_\varepsilon O^2_{n,n}(A_{\text{ét}})_{R_{\text{ét}}}$ to the pullback

$$P_n = B_\varepsilon O^2_{n,n}(\mathbb{F}_{\text{ét}})_{R_{\text{ét}}} \times_{B_\varepsilon O^2_{n,n}(*)_{R_{\text{ét}}}} B_\varepsilon O^2_{n,n}(W)_{R_{\text{ét}}}$$

factors as

$$B_\varepsilon O^2_{n,n}(A_{\text{ét}})_{R_{\text{ét}}} \xrightarrow{\alpha_n^*} B_\varepsilon O^2_{n,n}(M)_{R_{\text{ét}}} \xrightarrow{g_n} P_n.$$

where α_n^* is the map induced by α and g_n is the map induced by the maps from $B_\varepsilon O^2_{n,n}(M)_{R_{\text{ét}}}$ to the two corners of the square. The map g_n is a weak equivalence because the functor $B_\varepsilon O^2(-)_{R_{\text{ét}}}$ commutes up to homotopy with finite colimits. After taking the colimit over n we get the desired result by Corollary 7.4.

2) Because the maps from $*$ and W to $R_{\text{ét}}$ are nullhomotopic, the mapping spaces are weakly equivalent to spaces of maps into the homotopy fiber F_n. This can be seen e.g. by considering the spectral sequence (11) for W and $*$. The weak equivalence $F_n \simeq B_\varepsilon O^2_{n,n}(\mathbb{C})$ gives weak equivalences $(B_\varepsilon O^2_{n,n}(\mathbb{C}))^W \simeq B_\varepsilon O^2_{n,n}(W)_{R_{\text{ét}}}$ and $B_\varepsilon O^2_{n,n}(\mathbb{C}) \simeq B_\varepsilon O^2_{n,n}(*)_{R_{\text{ét}}}$ and these are natural in n. We get the desired weak equivalences after taking the colimit over n.

□

The evaluation map $ev_* : (B_\varepsilon O^2(\mathbb{C}))^W \to B_\varepsilon O^2(\mathbb{C})$ is a fibration with fiber $\prod_{i=1}^c \Omega B_\varepsilon O^2(\mathbb{C})$. It is split by the map sending a point $x \in B_\varepsilon O^2(\mathbb{C})$ to the constant map $W \to B_\varepsilon O^2(\mathbb{C})$ at x. Since the square (13) is homotopy cartesian, the homotopy fiber of the map $B_\varepsilon O^2(A_{\text{ét}})_{R_{\text{ét}}} \to B_\varepsilon O^2(\mathbb{F}_{\text{ét}})_{R_{\text{ét}}}$ is weakly equivalent to $\Omega(B_\varepsilon O^2(\mathbb{C}))^c$. The base and the fiber of the fiber sequence

$$\textstyle\prod_{i=1}^c \Omega B_\varepsilon O^2(\mathbb{C}) \longrightarrow (B_\varepsilon O^2(\mathbb{C}))^W \longrightarrow B_\varepsilon O^2(\mathbb{C})$$

are both infinite loop spaces, so, since it has a section this sequence is weakly equivalent to the split fiber sequence

$$\textstyle\prod_{i=1}^c \Omega B_\varepsilon O^2(\mathbb{C}) \longrightarrow \left(\prod_{i=1}^c \Omega B_\varepsilon O^2(\mathbb{C})\right) \times B_\varepsilon O^2(\mathbb{C}) \longrightarrow B_\varepsilon O^2(\mathbb{C}).$$

As a consequence the fiber sequence

$$\textstyle\prod_{i=1}^c \Omega B_\varepsilon O^2(\mathbb{C}) \longrightarrow B_\varepsilon O^2(A_{\text{ét}})_{R_{\text{ét}}} \longrightarrow B_\varepsilon O^2(\mathbb{F}_{\text{ét}})_{R_{\text{ét}}} \tag{14}$$

splits.

For a ring S with $2 \in S^\times$ the 2-adic étale ε-hermitian K-theory space ${}_\varepsilon KQ^{\text{ét}}(S)_2$ fits into an Ω-spectrum ${}_\varepsilon\mathcal{KQ}^{\text{ét}}(S)_2$ by the same construction and arguments as for étale K-theory (see sections 3 and 4 of [16]). The comparison map for ordinary and étale ε-hermitian K–theory extends to a map of spectra ${}_\varepsilon\mathcal{KQ}(S) \to {}_\varepsilon\mathcal{KQ}^{\text{ét}}(S)_2$. For algebraic K-theory Dwyer and Friedlander showed [16, 8.6] that the comparison map for $\mathbb{F}$ is a 2-adic equivalence on -1-connective covers. The arguments given do not generalize to ε-hermitian K-theory, so in stead we use arguments from [11]. I thank my adviser Paul Arne Østvær for pointing out to me the following two lemmas and their proofs.

Lemma 7.6: *Let $\mathbb{F}$ be a finite field of characteristic different from 2. Then the comparison map ${}_\varepsilon\mathcal{KQ}(\mathbb{F}) \to {}_\varepsilon\mathcal{KQ}^{\text{ét}}(\mathbb{F})_2$ is a 2-adic equivalence on -1-connective covers*

Proof. By [10, 7.4] there is a homotopy equivalence ${}_\varepsilon KQ(\mathbb{F}) \simeq K(\mathbb{F})^{h\mathbb{Z}/2}$ where the latter space is the homotopy fix points of the action of $\mathbb{Z}/2$ on $GL(\mathbb{F})$ by conjugation by the matrix J_ε of Example 2.21. The same argument as in the proof of [11, Th.7.6] now finishes the proof. □

Lemma 7.7: *The comparison map ${}_\varepsilon\mathcal{KQ}(A) \to {}_\varepsilon\mathcal{KQ}^{\text{ét}}(A)_2$ is a 2-adic equivalence on 0-connective covers.*

Proof. Lemma 7.5 of [11] says that if the comparison map is an equivalence on 0-connective covers for the residue fields $k(x)$ of all points $x \in Spec(A)$ then it is also an equivalence on 0-connective covers for A. Lemma 7.6 takes care of the closed points and [11, 7.6] takes care of the generic point. It follows that the map is a 2-adic equivalence. □

For a spectrum $\mathcal{E}$ the spectrum $\mathcal{E}^{(n)}$ denotes an n-connective cover of $\mathcal{E}$. From the split fiber sequence (14) and Lemma 7.6 we conclude the following:

Theorem 7.8: *Let $\varepsilon \in \{\pm 1\}$ and $A = \mathbb{Z}[\frac{1}{2}, \zeta_{2^\nu}]$ and $\mathbb{F} = \mathbb{F}_p$ for some prime p which is congruent to 5 mod 8 and is totally split in A. Then there is a split fiber sequence of 0-connective spectra*

$$\bigvee_{i=1}^{c} \Omega({}_{\varepsilon}\mathcal{KQ}^{top}(\mathbb{C})_2)^{(0)} \longrightarrow {}_{\varepsilon}\mathcal{KQ}(A)_2^{(0)} \longrightarrow {}_{\varepsilon}\mathcal{KQ}(\mathbb{F})_2^{(0)}.$$

The subscript 2's denote 2-completion.

Corollary 7.9: *For $\varepsilon \in \{\pm 1\}$ there is a 2-adic equivalence of 0-connective spectra*

$${}_{\varepsilon}\mathcal{KQ}(\mathbb{Z}[\tfrac{1}{2}, \sqrt{-1}])^{(0)} \simeq_2 {}_{\varepsilon}\mathcal{KQ}(\mathbb{F}_5)^{(0)} \vee (\Omega_{\varepsilon}\mathcal{KQ}^{top}(\mathbb{C}))^{(0)}.$$

In Example 3.7 we computed the 0-th ε-hermitian groups of $\mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ and Corollary 7.9 lets us compute the higher 2-completed groups using the table in [19, Th.1.7] for the groups ${}_{\varepsilon}KQ_n(\mathbb{F}_5)$ and the table [30, Th.5.19,III] for the groups ${}_{\varepsilon}KQ^{top}_{n+1}(\mathbb{C})$.

Theorem 7.10: *Let $t(n)$ be the largest power of 2 that divides $5^{\frac{n+1}{2}} - 1$ and let δ_{n0} denote the Kronecker symbol. Then, for $\varepsilon \in \{\pm 1\}$, the 2-completed ε-hermitian K-groups of $\mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ are given up to isomorphism by the following table:*

$n \mod 8$	${}_1KQ_n(\mathbb{Z}[\frac{1}{2}, \sqrt{-1}])$	${}_{-1}KQ_n(\mathbb{Z}[\frac{1}{2}, \sqrt{-1}])$
0	$\delta_{n0}\mathbb{Z}_2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\delta_{n0}\mathbb{Z}_2$
1	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	0
2	$\mathbb{Z}/2$	0
3	$\mathbb{Z}/2^{t(n)} \oplus \mathbb{Z}_2$	$\mathbb{Z}/2^{t(n)} \oplus \mathbb{Z}_2$
4	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
5	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$
6	0	$\mathbb{Z}/2$
7	$\mathbb{Z}/2^{t(n)} \oplus \mathbb{Z}_2$	$\mathbb{Z}/2^{t(n)} \oplus \mathbb{Z}_2$

A Appendix: Pro-categories and pro-homotopy theory

Definition A.1. [1, A.1.2.] A category $\mathcal{I}$ is called filtering, or filtered, if it satisfies the following two conditions:

1. For each pair of objects i and i' in $\mathcal{I}$ there is an object i'' and a diagram $i \to i'' \leftarrow i'$ in $\mathcal{I}$.
2. For each pair of morphisms $i \rightrightarrows i'$ there is an object i'' and a morphism $i' \to i''$ such that the two composites of $i \rightrightarrows i' \to i''$ are equal.

We will assume that all our filtering categories are small.

Filtering categories frequently show up in the form of indexing categories for diagrams. Diagrams $F : \mathcal{I} \to \mathscr{C}$ where $\mathcal{I}$ is filtering often behave well with respect to colimits.

Example A.2. [56, 2.6.15] Let R be a ring and $R - Mod$ the category of left R-modules. Further, let $\mathcal{I}$ be a filtering category. Then the functor $\operatorname{colim} : (R - Mod)^{\mathcal{I}} \to R - Mod$ is exact.

It is also interesting to consider categories whose opposite categories are filtering. We call such categories *cofiltering*.

Definition A.3. A pro-object in a category $\mathscr{C}$ is a functor $X : \mathcal{I}^{op} \to \mathscr{C}$ where $\mathcal{I}$ is a filtering category. We think of $\mathcal{I}$ as an index category, and hence write X_i for the value of X at $i \in \mathcal{I}$. Equivalently we can define a pro-object as a functor $X : \mathcal{I} \to \mathscr{C}$ where $\mathcal{I}$ is cofiltering.

Let $\mathrm{pro}{-}\mathscr{C}$ be the category whose objects are pro-objects in $\mathscr{C}$, and whose morphisms are given by

$$\hom_{\mathrm{pro}-\mathscr{C}}(X, Y) = \lim_{\mathcal{I}} \operatorname*{colim}_{\mathcal{J}} \mathscr{C}(X, Y)$$

for pro-objects $X : \mathcal{I} \to \mathscr{C}$ and $Y : \mathcal{J} \to \mathscr{C}$ in $\mathscr{C}$. Note that in general $\mathcal{I} \neq \mathcal{J}$.

Remark A.4. Considering objects of $\mathscr{C}$ as functors $\{*\} \to \mathscr{C}$ defines a full imbedding of $\mathscr{C}$ in $\mathrm{pro}{-}\mathscr{C}$.

Example A.5. Let G be a group. The finite quotient groups of G form a pro-object $\hat{G}$ in the category of finite groups. It is indexed by $\mathcal{N}(G)$, the category whose objects are of normal subgroups of finite index in G and whose morphisms inclusions of groups. Let

$$\hat{G} : \mathcal{N}(G)^{op} \to FinGrp$$

be the functor which sends a subgroup $N \in obj(\mathcal{N}(G))$ to G/N, and an inclusion $N \hookrightarrow N'$ to the induced map $G/N' \to G/N$. In [1] $\hat{G}$ is called the pro-finite completion of G, this term is usually used only for the group obtained by taking the limit of $\hat{G}$.

The category $FinGrP$ of finite groups is a full subcategory of the category Grp of all groups. If we take a group G then we can restrict the functor

$$h^G = hom_{Grp}(G, -) : Grp \to Set$$

to $FinGrp$ and ask whether it is representable. Since $FinGrp$ is a full subcategory this will occur if and only if G is finite. However, any map from a group G to a finite group H must factor through a finite quotient of G, so there is a natural isomorphism

$$h^G(H) \cong \operatorname*{colim}_{\mathcal{N}(G)} hom_{FinGrp}(\hat{G}, H).$$

We say that the functor $h^G : FinGrP \to Set$ is pro-representable [1, A.2.4].

Example A.6. Let G a group and p a prime number. Just as for finite groups the category Fin_pGrp of finite p-groups is a full subcategory of Grp and the finite p-order quotient groups of G form pro-object $\hat{G}_p$ in Fin_pGrp.

Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ a functor. Then there is an obvious induced functor $\text{pro}{-}F : \text{pro}{-}\mathscr{C} \to \text{pro}{-}\mathscr{D}$.

Definition A.7. Let $\mathcal{I}$ and $\mathcal{J}$ be cofiltering categories and let $X : \mathcal{I} \to \mathscr{C}$ and $Y : \mathcal{J} \to \mathscr{C}$ be pro-objects of $\mathscr{C}$. A strict map from X to Y is a pair (F, φ) where F is a functor $F : \mathcal{J} \to \mathcal{I}$ and φ is a natural transformation $\varphi : X \circ F \implies Y$. In diagrams:

$$\begin{tikzcd} \mathcal{J} \arrow[rr, "Y"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Uparrow \varphi"] & \mathscr{C} \\ & \mathcal{I} \arrow[ur, "X"'] & \end{tikzcd}$$

A strict map from X to Y clearly determines a map from X to Y in $\text{pro}{-}\mathscr{C}$.

Example A.8. Let $i : \mathcal{J} \hookrightarrow \mathcal{I}$ be an embedding of cofiltering categories and $X : \mathcal{I} \to \mathscr{C}$ a pro-object of $\mathscr{C}$. Then the identity transformation $(i, id : X \circ i \to X \circ i)$ is a strict map.

Definition A.9. [1, A.1.5] Let $\mathcal{I}$ be a filtering category and $\mathcal{J}$ any category. A functor $F : \mathcal{I} \to \mathcal{J}$ is called cofinal if it satisfies the following conditions:

- For every object j of $\mathcal{J}$ there is an object i of $\mathcal{I}$ and a map $j \to F(i)$.
- For objects i of $\mathcal{I}$ and j of $\mathcal{J}$ and maps $j \rightrightarrows F(i)$ there is a map $i \to i'$ in $\mathcal{I}$ such that the composed maps $j \rightrightarrows F(i')$ are equal.

With the hypotheses of the definition one can easily verify that also $\mathcal{J}$ is filtering. A subcategory $\mathcal{I}$ of a filtering category $\mathcal{J}$ is called cofinal if the inclusion functor $i : \mathcal{I} \hookrightarrow \mathcal{J}$ is cofinal. If $F : \mathcal{I} \to \mathcal{J}$ is a cofinal functor and $G : \mathcal{J} \to Set$ is a functor, the induced map on colimits $\operatorname{colim} FG \to \operatorname{colim} G$ is an isomorphism [1, A.1.8]. It follows [1, A.2.5] that any a pro-object $X : \mathcal{J}^{op} \to \mathscr{C}$ is isomorphic (in $\text{pro}{-}\mathscr{C}$) to its restriction $X \circ F : \mathcal{I}^{op} \to \mathscr{C}$.

The following lemma is useful when we want to prove things about pro-objects.

Lemma A.10: *Let $f : X \to Y$ be a map in $\mathrm{pro}-\mathscr{C}$ and let $[1]$ denote the category $0 \to 1$ with two objects and one morphism between them. Then there is a pro-object in the arrow category $\hat{f} : \mathcal{I} \to \mathscr{C}^{[1]}$ such that the pro-objects $\hat{f}_0$ and $\hat{f}_1$ we get by evaluation $\hat{f}$ on 0 and 1 are isomorphic to X and Y respectively and the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\cong"'] & Y \arrow[d, "\cong"] \\ \hat{f}_0 \arrow[r] & \hat{f}_1. \end{tikzcd}$$

The lower map is the natural transformation from $\hat{f}_0$ to $\hat{f}_1$ arising from the map $0 \to 1$.

Proof. [1, 160-161] □

By induction on the number of vertices this can be extended to finite diagrams without non-trivial loops.

We finish the appendix by defining homotopy invariants for pro-simplicial sets.

Definition A.11. Let X be a pro-object, indexed by $\mathcal{I}$, in the category $SSet_*$ of based simplicial sets. For $n \geq 2$ the n-th homotopy group $\pi_n(X)$ of X is defined to be the pro-abelian group given by the composition of functors

$$\mathcal{I}^{op} \xrightarrow{X} SSet_* \xrightarrow{|-|} CW_* \xrightarrow{\pi_n} Ab$$

where $|-|$ denotes geometric realization (see [21, I.2]). Likewise, for $n = 1$ and $n = 0$ the functor π_n gives pro-groups or pro-pointed sets respectively.

A map $X \to Y$ in $\mathrm{pro}-SSet$ is called a weak equivalence of pro-spaces if the induced maps on π_n's are isomorphisms in the appropriate pro-categories.

Definition A.12. Let X be a pro-simplicial set indexed by $\mathcal{I}$ and let A be an abelian group.

- The n-th homology group $H_n(X; A)$ of X with coefficients in A is defined to be the pro-abelian group defined by the composition

$$\mathcal{I}^{op} \xrightarrow{X} SSet_* \xrightarrow{H_n(-;A)} Ab\ .$$

- The n-th cohomology group of X with coefficients in A is the abelian group defined by

$$H_n(X) = \operatorname*{colim}_{i \in obj\mathcal{I}} H_n(X_i; A).$$

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